Finite Element Method for Continuum Solid Mechanics

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Outline

- Continuum Solid Mechanics : an Overview
- From Strong to Weak Form
- Finite Element Method
  - Discretization
  - Interpolation
  - Differentiation
  - Matrix assembly
  - Boundary conditions
  - Integration
- Mesh Adaptivity and Convergence
- Beyond Linear Problems: Illustrations

Continuum Solid Mechanics:
a Brief Review

Notations

- Scalars: a, α
- Vectors: $\underline{a}, \underline{\lambda}$ *(for a selected basis they are equivalent to* a_i, λ_j*)*
- Second order tensors: $\underline{\underline{A}}, \underline{\underline{\sigma}}$ $(\ldots A_{ij}, \sigma_{ij})$
- Scalar, vector, tensor products: $\underline{a} \cdot \underline{n}$, $\underline{\tau}_1 \times \underline{\tau}_2$, $\underline{e}_i \otimes \underline{e}_j$
- Double contraction: $\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \sim \sigma_{ij}\varepsilon_{ij}$

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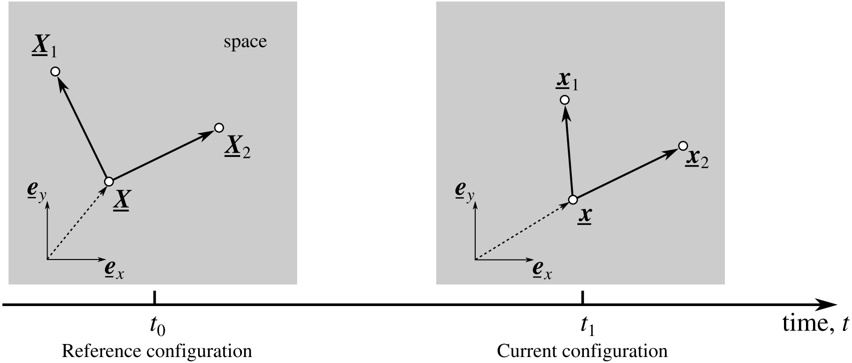
- Gradient operator: $\nabla\alpha, \nabla\underline{u}$, $\nabla\alpha = \dfrac{\partial\alpha}{\partial\underline{X}}$, $\nabla\underline{u} = \dfrac{\partial\underline{u}}{\partial\underline{X}}$

  for the Cartesian coordinates: $\nabla\underline{u} \sim \partial u_i/\partial X_j = u_{i,j}$
- Divergence operator: $\nabla \cdot \underline{\underline{\sigma}}$

  for the Cartesian coordiantes: $\nabla \cdot \underline{\underline{\sigma}} = \dfrac{\partial\underline{\underline{\sigma}}}{\partial X_j} \cdot \underline{e}_j = \sigma_{ij,j}\underline{e}_i \sim \sigma_{ij,j}$

Deformable medium

- Deformation in time t
- Reference configuration at $t = t_0$, $\underline{X}$ and current configuration at $t = t_1$, $\underline{x}(\underline{X}, t)$
- Lagrangian description, follow material points $\underline{X} = \underline{x}(t = t_0)$
- Displacement vector is $\underline{u} = \underline{x} - \underline{X}$



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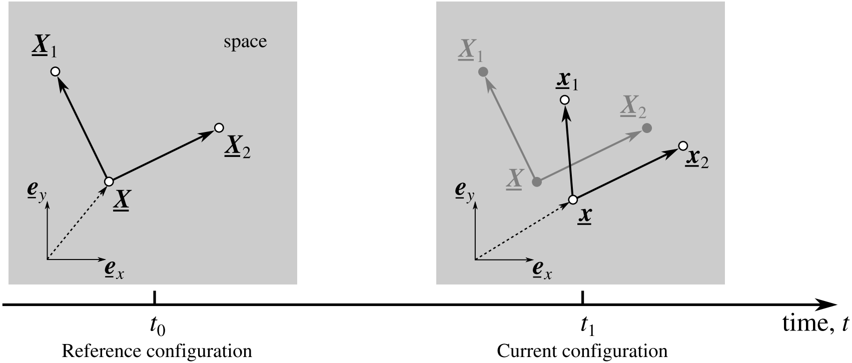

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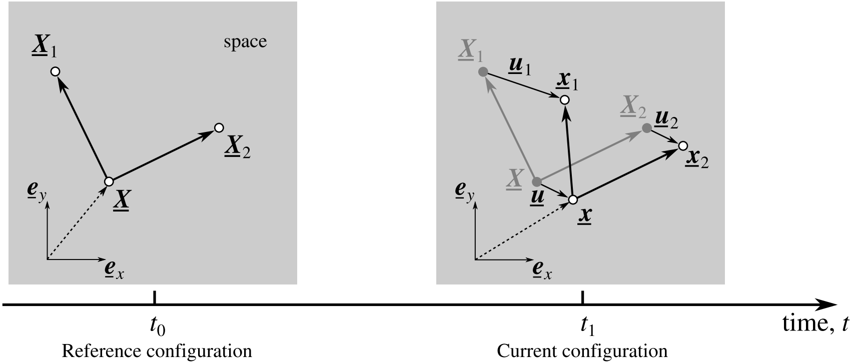
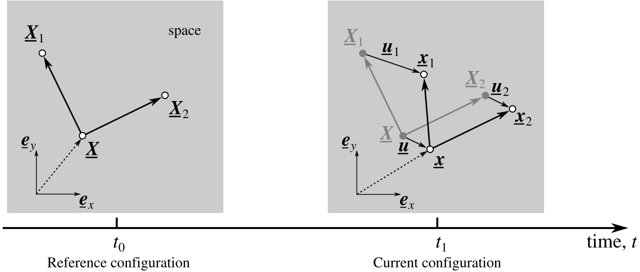


Deformation tensor

- Transformation gradient $\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{X}} = \frac{\partial\left(\underline{X} + \underline{u}\right)}{\partial \underline{X}} = \underline{\underline{I}} + \frac{\partial \underline{u}}{\partial \underline{X}} = \underline{\underline{I}} + \underline{\underline{H}}$
- Cauchy-Green right tensor $\underline{\underline{C}} = \underline{\underline{F}}^{\top} \cdot \underline{\underline{F}}$
- Green-Lagrange deformation tensor $\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{C}} - \underline{\underline{I}}\right) = \underline{\underline{H}}^{S} + \frac{1}{2}\underline{\underline{H}}^{\top} \cdot \underline{\underline{H}}$
- For $H_{ij} \ll 1$, $\underline{\underline{E}} \approx \underline{\underline{H}}^{S}$ and we obtain a tensor of small deformations

$$\underline{\underline{\varepsilon}} = \underline{\underline{H}}^{S} = \frac{1}{2}\left[\frac{\partial \underline{u}}{\partial \underline{X}} + \left(\frac{\partial \underline{u}}{\partial \underline{X}}\right)^{\top}\right] = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^{\top}\right)$$



Stress tensor and Hooke's law

- Hooke's law in uniaxial test:

$$\boxed{\sigma_{xx} = E\varepsilon_{xx}}$$

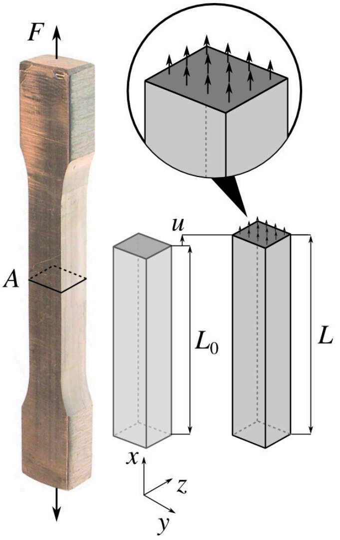
$$F = ku \quad \Leftrightarrow \quad \sigma_{xx}A = \frac{EA}{L_0}u = EA\frac{L-L_0}{L_0}$$

- In general case stress and strain are related through a linear operator (fourth-order elasticity tensor ${}^4\underline{\underline{C}}$):

$$\boxed{\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : \underline{\underline{\varepsilon}}}$$

- Inversely the strain can be found through a stiffness tensor ${}^4\underline{\underline{S}}$:

$$\boxed{\underline{\underline{\varepsilon}} = {}^4\underline{\underline{S}} : \underline{\underline{\sigma}}}$$



Hooke's law for **isotropic solids**: stress

- In the case of an **isotropic material**, Hooke's law reduces to:

$$\boxed{\underline{\underline{\sigma}} = \lambda \operatorname{tr}(\underline{\underline{\varepsilon}})\underline{\underline{I}} + 2\mu\underline{\underline{\varepsilon}},}$$

with λ, μ being Lamé coefficients:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

with Young's modulus E and Poisson's ratio ν.

- In the component form it reads:

$$\sigma_{ij} = \lambda(\varepsilon_{kk})\delta_{ij} + 2\mu\varepsilon_{ij}$$

- In the matrix form:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = 2\mu \begin{bmatrix} \lambda \operatorname{tr}(\underline{\underline{\varepsilon}})/(2\mu) + \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \lambda \operatorname{tr}(\underline{\underline{\varepsilon}})/(2\mu) + \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \lambda \operatorname{tr}(\underline{\underline{\varepsilon}})/(2\mu) + \varepsilon_{33} \end{bmatrix}$$

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Hooke's law for isotropic solids: strain

- Strain as a function of stress:

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}}.$$

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$$= \frac{1}{E}\begin{bmatrix} \sigma_{11} - \nu(\sigma_{22}+\sigma_{33}) & (1+\nu)\sigma_{12} & (1+\nu)\sigma_{13} \\ (1+\nu)\sigma_{12} & \sigma_{22} - \nu(\sigma_{11}+\sigma_{33}) & (1+\nu)\sigma_{23} \\ (1+\nu)\sigma_{13} & (1+\nu)\sigma_{23} & \sigma_{33} - \nu(\sigma_{11}+\sigma_{22}) \end{bmatrix}$$

Equilibrium of an infinitesimal element

- Infinitesimal strain tensor is symmetric and satisfies the compatibility conditions*:

$$\nabla \times \left(\nabla \times \underline{\underline{\varepsilon}}\right) = 0$$

- Stress tensor $\underline{\underline{\sigma}}$ should ensure equilibrium of any infinitesimal element**:

  Force balance: $\int_S \underline{n} \cdot \underline{\underline{\sigma}} \, dS = 0$

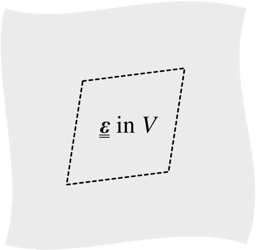
  Momentum balance: $\int_S \underline{r} \times (\underline{n} \cdot \underline{\underline{\sigma}}) \, dS = 0$

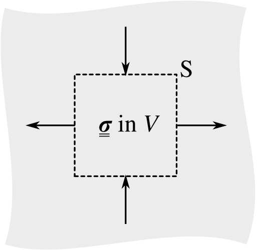
- Following the divergence theorem:

$$\int_S \underline{n} \cdot \underline{\underline{\sigma}} \, dS = \int_V \nabla \cdot \underline{\underline{\sigma}} \, dV = 0$$

Since the volume V can be arbitrary chosen, then

$\boxed{\nabla \cdot \underline{\underline{\sigma}} = 0}$ everywhere in V.





*In the case of a simply-connected solid.
**In the absence of volumetric forces.

Equilibrium of an infinitesimal element II

- Second Newton's law:

$$m\underline{\ddot{u}} = \underline{f} \quad \Rightarrow \quad \rho\underline{\ddot{u}} = \frac{1}{V}\underline{f}$$

- In presence of volumetric forces with density $\underline{f}_V$, the total force is given by:

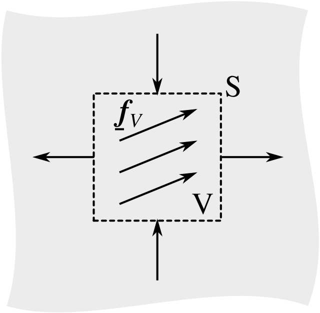
$$\underline{f} = \int_V \underline{f}_V \, dV + \int_S \underline{n} \cdot \underline{\underline{\sigma}} \, dS$$

- Then using the second Newton's law and the divergence theorem:

$$\int_V \left(\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V \right) dV = \int_V \rho\underline{\ddot{u}} \, dV$$

- Since it holds for an arbitrary V, then in every point of V:

$$\boxed{\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V = \rho\underline{\ddot{u}}}$$



Equilibrium of an infinitesimal element II

- Equilibrium (3 equations):

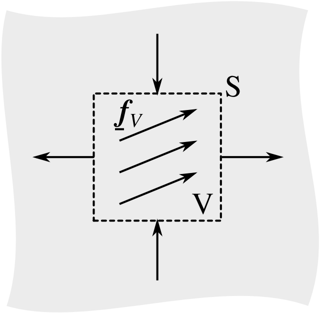
$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V = \rho \underline{\ddot{u}}$$

- In component form (Cartesian coordinates)*:

$$\sigma_{ij,j} + f_{Vi} = \rho \ddot{u}_i,$$

- Explicitly:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_{V_x} = \rho \ddot{u}_x$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_{V_y} = \rho \ddot{u}_y$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_{V_z} = \rho \ddot{u}_z$$



* The following notation is used $y_{i,j} = \frac{\partial y_i}{\partial x_j}$

Deformable solid and boundary conditions

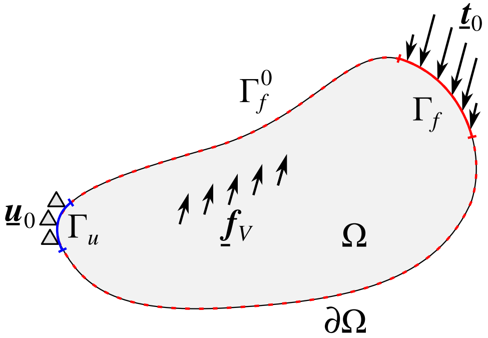
Notations:

- Consider a solid Ω with boundary $\partial\Omega$
- Boundary is split into Γ_u and Γ_f: $\partial\Omega = \Gamma_u \cup \Gamma_f$
- At Γ_u displacements $\underline{u}_0(t, \underline{X})$ are prescribed (Dirichlet boundary conditions [BC]):

  $$\underline{u} = \underline{u}_0 \text{ at } \Gamma_u$$

- At Γ_f tractions $\underline{t}_0(t, \underline{X})$ are prescribed (Neumann BC):

  $$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 \text{ at } \Gamma_f$$
  $$\underline{\underline{\sigma}} \cdot \underline{n} = 0 \text{ at } \Gamma_f^0$$



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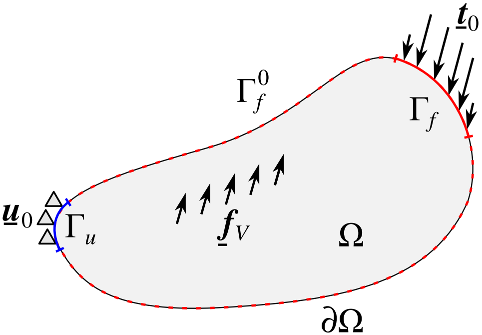


Remarks:

- on the same boundary both BCs can be prescribed if they are orthogonal one to each other, i.e. $\underline{u}_0 \cdot \underline{t}_0 = 0$ (*ex.:* friction);
- a relationship between these BCs can be prescribed (Robin BC): $\underline{u}_0 = \underline{U} - s\underline{t}_0$ (*ex.:* Winkler's foundation).

Elastic and static problem set-up

- Equilibrium in the absence of inertial forces

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V = 0 \quad (*)$$

- Consistutive relation:

$$\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : \underline{\underline{\varepsilon}}$$

- Strain tensor:

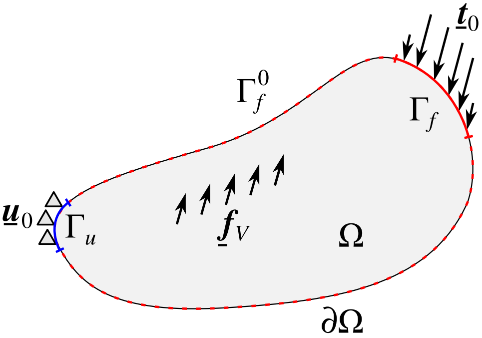
$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^\top\right)$$

- Boundary conditions:

  $\underline{u} = \underline{u}_0$ at Γ_u (Dirichlet or essential BC)

  $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0$ at Γ_f (Neumann or natural BC)

  $\underline{\underline{\sigma}} \cdot \underline{n} = 0$ at Γ_f^0 (Trivial Neumann BC)



- **Problem:**
  find such field $\underline{u}$ in Ω that satisfies equilibrium Eq. $(*)$ and boundary conditions.

From Strong to Weak Form

Equilibrium: from strong to weak form

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- Boundary conditions (BC):

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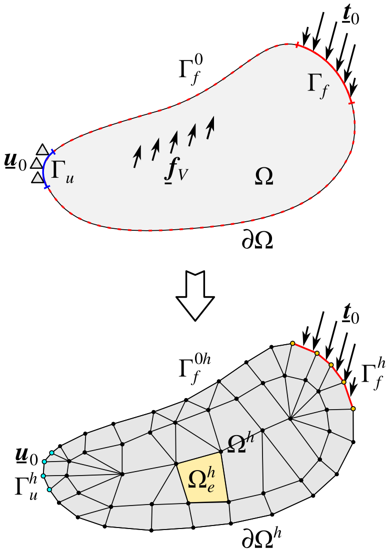
$\underline{\underline{\sigma}} \cdot \underline{n} = 0$ at Γ_f^0



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$$\int_{\Omega} (\nabla \cdot \underline{\underline{\sigma}}) \cdot \underline{v} \, dV + \int_{\Omega} \underline{f}_V \cdot \underline{v} \, dV = 0$$

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- Since $\int_\Omega \nabla \cdot (\underline{\underline{\sigma}} \cdot \underline{v}) \, dV = \int_\Omega (\nabla \cdot \underline{\underline{\sigma}}) \cdot \underline{v} \, dV + \int_\Omega \underline{\underline{\sigma}} : (\nabla \underline{v}) \, dV$

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$$\boxed{\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{v}\, dS - \int_\Omega \underline{\underline{\sigma}} : (\nabla \underline{v})\, dV + \int_\Omega \underline{f}_V \cdot \underline{v}\, dV = 0}$$

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- If we select virtual vector field $\underline{v} = \delta \underline{u}$ as virtual displacements vanishing at Γ_u ($\int_{\Gamma_u} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{u} \, dS = 0$) and denote $\delta \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}(\delta \underline{u})$, we obtain:

$$\int_{\Gamma_f} \underline{t}_0 \cdot \delta \underline{u} \, dS - \int_\Omega \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}} \, dV + \int_\Omega \underline{f}_V \cdot \delta \underline{u} \, dV = 0$$

- This variational formulation is called the *principle of virtual work*.

Weak form

- Work of imposed surface tractions on *virtual* displacements $= \frac{1}{2}\, \underline{t}_0 \cdot \delta \underline{u}$
- Work density of distributed volumetric forces $= \frac{1}{2}\, \underline{f}_V \cdot \delta \underline{u}$
- Corresponding virtual density of elastic energy $= \frac{1}{2}\, \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}}$

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- According to the principle of virtual work:

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- Equivalently

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- So we are in the framework of the *Lax-Milgram theorem* (continuity and coercivity could be easily shown).

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$$a(\underline{u}, \delta\underline{u}) = \int\limits_{\Omega} \frac{1}{2}(\nabla\underline{u} + (\nabla\underline{u})^\top) : \underline{\underline{C}} : \nabla\delta\underline{u}\, dV$$

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- For an isotropic material, the stress is: $\underline{\underline{\boldsymbol{\sigma}}} = \lambda \operatorname{tr}(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\boldsymbol{I}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}}$

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Finite Element Method

Main idea in a nutshell

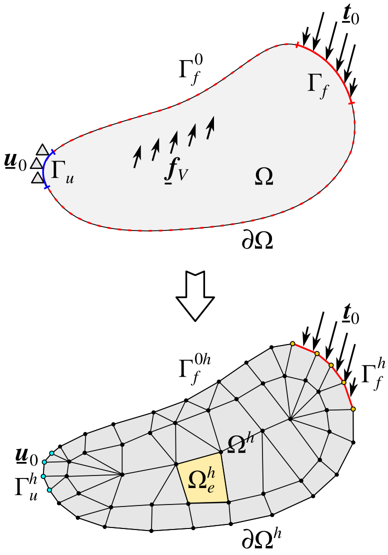
- Find displacements only at certain locations $\underline{u}_i(t)$ and interpolate in between

$$\underline{u}(\underline{X},t) = \sum N_i(\underline{X})\underline{u}_i(t)$$

- Virtual displacements are also interpolated in the same way

$$\delta\underline{u}(\underline{X}) = \sum N_i(\underline{X})\delta\underline{u}_i$$

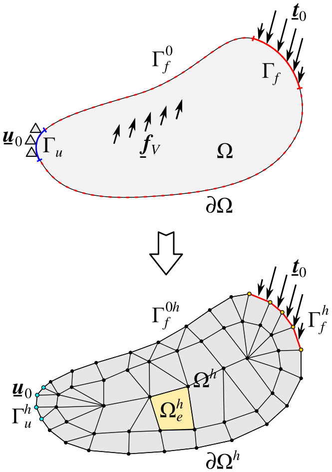
- Thus, we reduce the problem of dimension ∞ to a finite dimensional problem
- Weak formulation of equilibrium equations results in a linear system of equations...



Main idea

- From continuous to discrete problem
- Split solid into finite elements

  $\Omega \rightarrow \Omega^h$ with $\Omega^h = \sum_e \Omega_e^h$
- All quantities are associated with this discretization:
  $\underline{u} \rightarrow \underline{u}^h, \underline{\underline{\sigma}} \rightarrow \underline{\underline{\sigma}}^h, \Gamma_f \rightarrow \Gamma_f^h, \underline{t}_0 \rightarrow \underline{t}_0^h, \ldots$
- Search for $\underline{u}^h$ only in a finite number of points (nodes)
- Interpolate in between (within elements)



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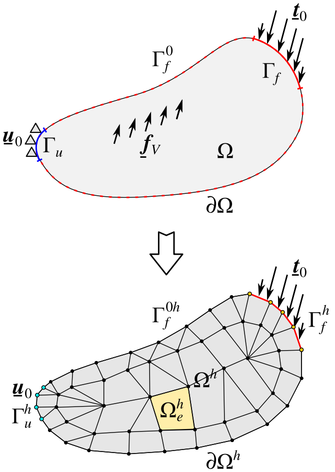


- Existence and uniqueness of the solution $\underline{u}^h_*$
- When discretization-size tends to zero $h \to 0$, convergence to the solution of the continuum problem: $\underline{u}^h_* \xrightarrow[h\to 0]{} \underline{u}_*$

Shape functions

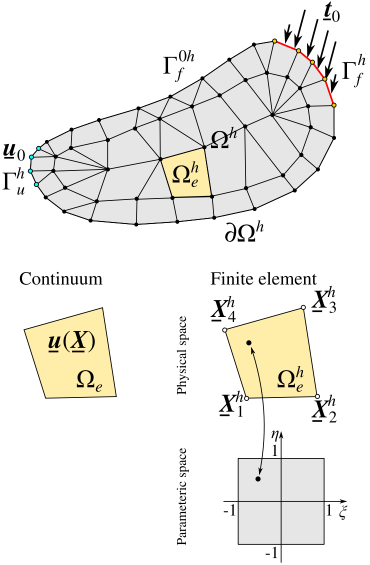
- Displacements are known at nodes: $\underline{\boldsymbol{u}}_i^h,\ i = 1, 4$
- We need to know them inside the element
- Parametrize the inside with parameters $\{\xi, \eta\} \in [-1, 1]$
- Use *interpolation* or *shape* functions $N_i(\xi, \eta)$ for position $\underline{\boldsymbol{X}}$

  $$\underline{\boldsymbol{X}}^h(\xi, \eta) = \sum_i \underline{\boldsymbol{X}}_i^h N_i(\xi, \eta)$$

  and displacement $\underline{\boldsymbol{u}}$:

  $$\underline{\boldsymbol{u}}^h(\xi, \eta) = \sum_i \underline{\boldsymbol{u}}_i^h N_i(\xi, \eta)$$

- If the same functions are used, then the element is called *isoparametric*
- Remark: Find $\{\xi, \eta\}$ from $\underline{\boldsymbol{X}}$ is not always straigthforward (may result in a system of non-linear equations)



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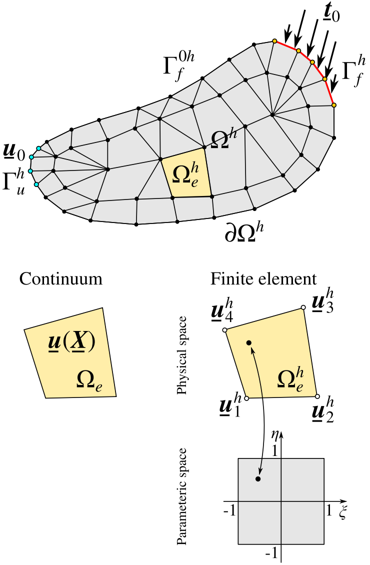


Shape functions II

Rules

- Node i has coordinates $\{\xi_i, \eta_i\}$
- Then $N_i(\xi_j, \eta_j) = \delta_{ij}$
- Partition of unity:

  $\forall \xi, \eta, : \sum_i N_i(\xi, \eta) = 1$

Types

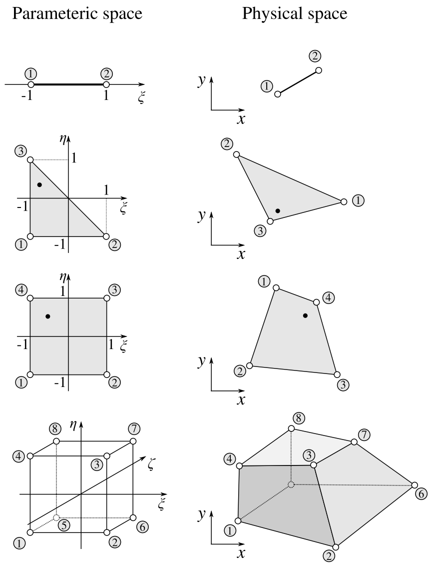
- Linear shape functions

  $$\frac{\partial N}{\partial \xi} = \text{const}$$

- Non-linear shape functions

  $$\frac{\partial N}{\partial \xi} = f(\xi)$$

- Linear elements vs quadratic elements
- Higher order elements



Shape functions III

Example: bar element

- Linear shape functions:

$$N_1(\xi) = \frac{1}{2}(1-\xi)$$

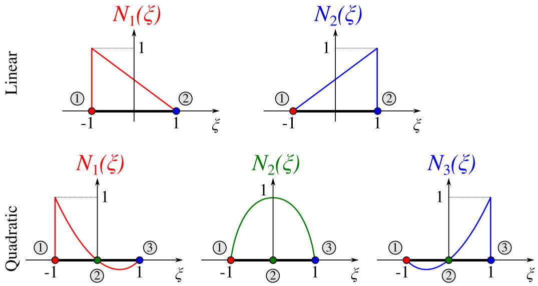
$$N_2(\xi) = \frac{1}{2}(1+\xi)$$

- Quadratic shape functions:

$$N_1(\xi) = \frac{1}{2}\xi(\xi-1)$$

$$N_2(\xi) = (1-\xi^2)$$

$$N_3(\xi) = \frac{1}{2}\xi(1+\xi)$$



Shape functions: vectors and matrices

- Displacement nodal vectors $\underline{u}_i = \underline{e}_x u_i^x + \underline{e}_y u_i^y$
- Array of nodal coordinates (size $\dim \cdot n$)

$$[X] = [x_1,\ y_1,\ x_2,\ y_2,\ \ldots\ x_n,\ y_n]_{2n}^\top$$

- Array of nodal displacements (size $\dim \cdot n$)

$$[\boldsymbol{u}] = [u_1^x,\ u_1^y,\ u_2^x,\ u_2^y,\ \ldots\ u_n^x,\ u_n^y]_{2n}^\top$$

- Arrays of shape functions (size $\dim \cdot n$)

$$[\boldsymbol{N_x}] = [N_1,\ 0,\ N_2,\ 0,\ \ldots\ N_n,\ 0]_{2n}^\top$$

$$[\boldsymbol{N_y}] = [0,\ N_1,\ 0,\ N_2,\ \ldots\ 0,\ N_n]_{2n}^\top$$

$$[\boldsymbol{N}] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \ldots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \ldots & 0 & N_n \end{bmatrix}_{2n\times\dim}^\top$$

- Then

$$x(\xi,\eta,t) = [\boldsymbol{N_x(\xi,\eta)}]^\top [\boldsymbol{X(t)}], \quad y(\xi,\eta,t) = [\boldsymbol{N_y(\xi,\eta)}]^\top [\boldsymbol{X(t)}]$$

$$u^x(\xi,\eta,t) = [\boldsymbol{N_x(\xi,\eta)}]^\top [\boldsymbol{u(t)}], \quad u^y(\xi,\eta,t) = [\boldsymbol{N_y(\xi,\eta)}]^\top [\boldsymbol{u(t)}]$$

Gradients and shape functions

- Need to evaluate gradients (spatial derivatives) like $\frac{\partial f}{\partial x}$
- But with shape functions $f = f(\xi, \eta)$
- Then $\dfrac{\partial f(\xi,\eta)}{\partial x} = \dfrac{\partial f}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial \eta}{\partial x}$
- However, in general we do not have $\xi = \xi(x, y)$ but rather $x = x(\xi, \eta)$
- Let's do it other way around

$$\begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = [\mathbf{J}] \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix}$$

- Matrix $[\mathbf{J}]$ is called Jacobian operator/matrix and enables to obtain

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = [\mathbf{J}]^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix}$$

Jacobian operator/matrix

- Jacobian operator/matrix:

$$[\mathbf{J}] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

- Using $x = [N_x]^\top [X], \quad y = [N_y]^\top [X]$ we get:

$$[\mathbf{J}] = \begin{bmatrix} [N_{x,\xi}]^\top [X] & [N_{y,\xi}]^\top [X] \\ [N_{x,\eta}]^\top [X] & [N_{y,\eta}]^\top [X] \end{bmatrix},$$

where $[N_{x,\xi}] = \left[\frac{\partial N_1}{\partial \xi}, \ 0, \ \frac{\partial N_2}{\partial \xi}, 0, \ \dots \frac{\partial N_n}{\partial \xi}, 0 \right]^\top$ etc.

- Then the inverse Jacobian is given by:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta} \begin{bmatrix} [N_{y,\eta}]^\top [X] & -[N_{y,\xi}]^\top [X] \\ -[N_{x,\eta}]^\top [X] & [N_{x,\xi}]^\top [X] \end{bmatrix},$$

with the determinant of the Jacobian matrix (or simply Jacobian):

$$\Delta = \det([\mathbf{J}]) = [X]^\top \left([N_{x,\xi}][N_{y,\eta}]^\top - [N_{y,\xi}][N_{x,\eta}]^\top \right) [X] \neq 0$$

Infinitesimal strain in 2D

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$$\nabla\underline{\boldsymbol{u}} = \frac{\partial\underline{\boldsymbol{u}}^h}{\partial x}\otimes\underline{\boldsymbol{e}}_x + \frac{\partial\underline{\boldsymbol{u}}^h}{\partial y}\otimes\underline{\boldsymbol{e}}_y = \frac{\partial u^x}{\partial x}\underline{\boldsymbol{e}}^x\otimes\underline{\boldsymbol{e}}^x + \frac{\partial u^x}{\partial y}\underline{\boldsymbol{e}}^x\otimes\underline{\boldsymbol{e}}^y + \frac{\partial u^y}{\partial x}\underline{\boldsymbol{e}}^y\otimes\underline{\boldsymbol{e}}^x + \frac{\partial u^y}{\partial y}\underline{\boldsymbol{e}}^y\otimes\underline{\boldsymbol{e}}^y$$

Infinitesimal strain in 2D

- Strain tensor: $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{\boldsymbol{u}} + (\nabla \underline{\boldsymbol{u}})^\top\right) \qquad (*)$
- Interpolated displacements: $u^x = [\boldsymbol{N}_x]^\top [\boldsymbol{u}], \quad u^y = [\boldsymbol{N}_y]^\top [\boldsymbol{u}]$
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$$(\nabla \underline{\boldsymbol{u}})^\top \sim \begin{bmatrix} \frac{\partial u^x}{\partial x} & \frac{\partial u^y}{\partial x} \\ \frac{\partial u^x}{\partial y} & \frac{\partial u^y}{\partial y} \end{bmatrix}$$

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Infinitesimal strain in 2D

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Infinitesimal strain in 2D

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Infinitesimal strain in 2D

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- Represent strain tensor as an array (Voigt notations):

$$\underline{\underline{\varepsilon}} \quad \Rightarrow \quad [\boldsymbol{E}] = \left[\varepsilon_{xx}, \quad \varepsilon_{yy}, \quad \gamma_{xy}\right]^\top, \quad \gamma_{xy} = 2\varepsilon_{xy}$$

Infinitesimal strain in 2D

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- Then

$$[\boldsymbol{E}] = \left[\frac{\partial u^x}{\partial x}, \quad \frac{\partial u^y}{\partial y}, \quad \frac{\partial u^y}{\partial x} + \frac{\partial u^x}{\partial y}\right]^\top$$

Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta}\begin{bmatrix} [N_{y,\eta}]^\top[X] & -[N_{y,\xi}]^\top[X] \\ -[N_{x,\eta}]^\top[X] & [N_{x,\xi}]^\top[X] \end{bmatrix}$$

Infinitesimal strain in 2D in matrix form

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- Then the strain components are

$$\varepsilon_{xx} = \left(J_{11}[\boldsymbol{N}_{x,\xi}] + J_{12}[\boldsymbol{N}_{x,\eta}]\right)^\top [\boldsymbol{u}] = \frac{1}{\Delta}\left([\boldsymbol{N}_{y,\eta}]^\top[\boldsymbol{X}][\boldsymbol{N}_{x,\xi}] - [\boldsymbol{N}_{y,\xi}]^\top[\boldsymbol{X}][\boldsymbol{N}_{x,\eta}]\right)^\top [\boldsymbol{u}]$$

Infinitesimal strain in 2D in matrix form

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Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta}\begin{bmatrix} [\boldsymbol{N}_{y,\eta}]^\top[\boldsymbol{X}] & -[\boldsymbol{N}_{y,\xi}]^\top[\boldsymbol{X}] \\ -[\boldsymbol{N}_{x,\eta}]^\top[\boldsymbol{X}] & [\boldsymbol{N}_{x,\xi}]^\top[\boldsymbol{X}] \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

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Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta}\begin{bmatrix} [\boldsymbol{N}_{y,\eta}]^\top[\boldsymbol{X}] & -[\boldsymbol{N}_{y,\xi}]^\top[\boldsymbol{X}] \\ -[\boldsymbol{N}_{x,\eta}]^\top[\boldsymbol{X}] & [\boldsymbol{N}_{x,\xi}]^\top[\boldsymbol{X}] \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

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$$\gamma_{xy} = \left(\frac{\partial u^x}{\partial y} + \frac{\partial u^y}{\partial x}\right) = \left(J_{11}[\boldsymbol{N}_{y,\xi}] + J_{12}[\boldsymbol{N}_{y,\eta}] + J_{21}[\boldsymbol{N}_{x,\xi}] + J_{22}[\boldsymbol{N}_{x,\eta}]\right)^\top [\boldsymbol{u}]$$

Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta}\begin{bmatrix} [\boldsymbol{N}_{y,\eta}]^\top[\boldsymbol{X}] & -[\boldsymbol{N}_{y,\xi}]^\top[\boldsymbol{X}] \\ -[\boldsymbol{N}_{x,\eta}]^\top[\boldsymbol{X}] & [\boldsymbol{N}_{x,\xi}]^\top[\boldsymbol{X}] \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

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$$\gamma_{xy} = \left(\frac{\partial u^x}{\partial y} + \frac{\partial u^y}{\partial x}\right) = \left(J_{11}[\boldsymbol{N}_{y,\xi}] + J_{12}[\boldsymbol{N}_{y,\eta}] + J_{21}[\boldsymbol{N}_{x,\xi}] + J_{22}[\boldsymbol{N}_{x,\eta}]\right)^\top[\boldsymbol{u}]$$

$$\gamma_{xy} = \frac{1}{\Delta}\left([\boldsymbol{N}_{y,\eta}]^\top[\boldsymbol{X}][\boldsymbol{N}_{y,\xi}] - [\boldsymbol{N}_{y,\xi}]^\top[\boldsymbol{X}][\boldsymbol{N}_{y,\eta}] - [\boldsymbol{N}_{x,\eta}]^\top[\boldsymbol{X}][\boldsymbol{N}_{x,\xi}] + [\boldsymbol{N}_{x,\xi}]^\top[\boldsymbol{X}][\boldsymbol{N}_{x,\eta}]\right)^\top[\boldsymbol{u}]$$

Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

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- Then the strain components are

$$\varepsilon_{xx} = \left(J_{11}[N_{x,\xi}] + J_{12}[N_{x,\eta}]\right)^\top [u] = \frac{1}{\Delta}\left([N_{y,\eta}]^\top[X][N_{x,\xi}] - [N_{y,\xi}]^\top[X][N_{x,\eta}]\right)^\top [u] = [B_1]^\top[u]$$

$$\varepsilon_{yy} = \left(J_{21}[N_{y,\xi}] + J_{22}[N_{y,\eta}]\right)^\top [u] = \frac{1}{\Delta}\left(-[N_{x,\eta}]^\top[X][N_{y,\xi}] + [N_{x,\xi}]^\top[X][N_{y,\eta}]\right)^\top [u] = [B_2]^\top[u]$$

$$\gamma_{xy} = \left(\frac{\partial u^x}{\partial y} + \frac{\partial u^y}{\partial x}\right) = \left(J_{11}[N_{y,\xi}] + J_{12}[N_{y,\eta}] + J_{21}[N_{x,\xi}] + J_{22}[N_{x,\eta}]\right)^\top [u]$$

$$\gamma_{xy} = \frac{1}{\Delta}\left([N_{y,\eta}]^\top[X][N_{y,\xi}] - [N_{y,\xi}]^\top[X][N_{y,\eta}] - [N_{x,\eta}]^\top[X][N_{x,\xi}] + [N_{x,\xi}]^\top[X][N_{x,\eta}]\right)^\top [u] = [B_3]^\top[u]$$

- Then

$$[E]_3 = [B]^\top_{3\times 2n}[u]_{2n}$$

- With $[B] = \left[[B_1]^\top, \quad [B_2]^\top, \quad [B_3]^\top\right]^\top$

Infinitesimal strain in 2D: example

- Consider a linear triangular element with shape functions:

  $N_1 = -\frac{1}{2}(\xi + \eta), \quad N_2 = \frac{1}{2}(1 + \xi), \quad N_3 = \frac{1}{2}(1 + \eta)$

- Their derivatives are given by:

  $N_{1,\xi} = -1/2, \quad N_{2,\xi} = 1/2, \quad N_{3,\xi} = 0$

  $N_{1,\eta} = -1/2, \quad N_{2,\eta} = 0, \quad N_{3,\eta} = 1/2$

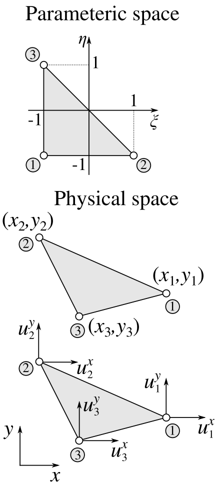
  $$\Delta = \frac{1}{4}((x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1))^*$$

- Then

  $$\varepsilon_{xx} = \frac{1}{4\Delta}\left[(y_3 - y_1)(u_2^x - u_1^x) - (y_2 - y_1)(u_3^x - u_1^x)\right]$$

  $$\varepsilon_{yy} = \frac{1}{4\Delta}\left[(x_2 - x_1)(u_3^y - u_1^y) - (x_3 - x_1)(u_2^y - u_1^y)\right]$$

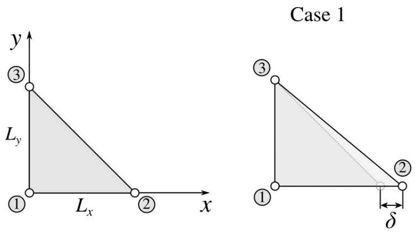
  $$\gamma_{xy} = \frac{1}{4\Delta}\left[(y_3 - y_1)(u_2^y - u_1^y) - (y_2 - y_1)(u_3^y - u_1^y) + (x_2 - x_1)(u_3^x - u_1^x) - (x_3 - x_1)(u_2^x - u_1^x)\right]$$



*Half of the area of the triangle.

Infinitesimal strain in 2D: example II

- Rectangular triangle $x_1 = x_3,\ y_1 = y_2, \Delta = L_x L_y/4$
- Case 1: pure tension/compression along OX iff $u_3^y = u_1^y, u_2^y = u_1^y, u_3^x = u_1^x$
  Ex.: $u_2^x = \delta:\quad \varepsilon_{xx} = \frac{1}{4\Delta}(y_3 - y_1)(u_2^x - u_1^x) = \delta/L_x,\quad \varepsilon_{yy} = \gamma_{xy} = 0$

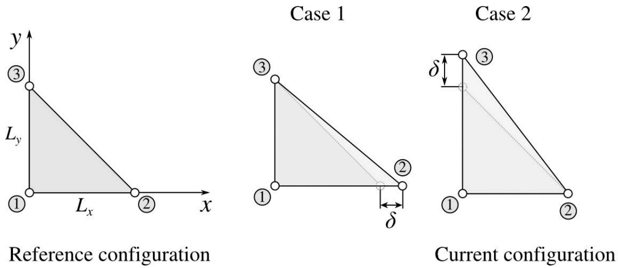


Reference configuration

Current configuration

Infinitesimal strain in 2D: example II

- Rectangular triangle $x_1 = x_3,\ y_1 = y_2,\ \Delta = L_x L_y/4$
- Case 1: pure tension/compression along OX iff $u^y_3 = u^y_1,\ u^y_2 = u^y_1,\ u^x_3 = u^x_1$
  Ex.: $u^x_2 = \delta :\quad \varepsilon_{xx} = \frac{1}{4\Delta}(y_3 - y_1)(u^x_2 - u^x_1) = \delta/L_x,\quad \varepsilon_{yy} = \gamma_{xy} = 0$
- Case 2: pure tension/compression along OY iff $u^x_2 = u^x_1,\ u^y_2 = u^y_1,\ u^x_3 = u^x_1$
  Ex.: $u^y_3 = \delta :\quad \varepsilon_{yy} = \frac{1}{4\Delta}(x_2 - x_1)(u^y_3 - u^y_1) = \delta/L_y,\quad \varepsilon_{xx} = \gamma_{xy} = 0$



Infinitesimal strain in 2D: example II

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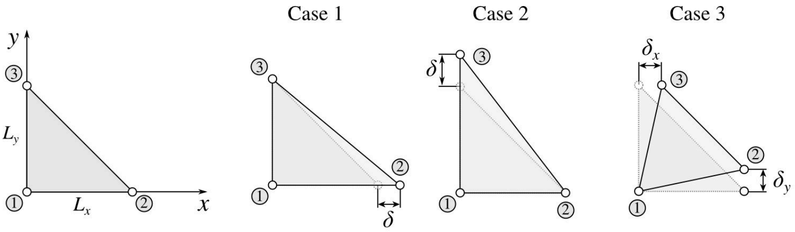
- Case 1: pure tension/compression along OX iff $u^y_3 = u^y_1,\ u^y_2 = u^y_1,\ u^x_3 = u^x_1$
  Ex.: $u^x_2 = \delta:\quad \varepsilon_{xx} = \frac{1}{4\Delta}(y_3 - y_1)(u^x_2 - u^x_1) = \delta/L_x,\quad \varepsilon_{yy} = \gamma_{xy} = 0$
- Case 2: pure tension/compression along OY iff $u^x_2 = u^x_1,\ u^y_2 = u^y_1,\ u^x_3 = u^x_1$
  Ex.: $u^y_3 = \delta:\quad \varepsilon_{yy} = \frac{1}{4\Delta}(x_2 - x_1)(u^y_3 - u^y_1) = \delta/L_y,\quad \varepsilon_{xx} = \gamma_{xy} = 0$
- Case 3: pure shear in XY iff $u^x_2 = u^x_1,\ u^y_3 = u^y_1$
  Ex.: $u^y_2 = \delta_y,\ u^x_3 = \delta_x:$

$$\gamma_{xy} = \frac{1}{4\Delta}\Big((y_3 - y_1)(u^y_2 - u^y_1) + (x_2 - x_1)(u^x_3 - u^x_1)\Big) = \frac{\delta_y}{L_x} + \frac{\delta_x}{L_y},\quad \varepsilon_{xx} = \varepsilon_{yy} = 0$$



Reference configuration

Current configuration

Stress tensor

- In linear elasticity, strain decomposition:

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}_{el} + \underline{\underline{\varepsilon}}_{th}$$

- With thermal strain field:

$$\underline{\underline{\varepsilon}}_{th} = \alpha(T - T_0)\underline{\underline{I}}$$

Stress tensor

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- With thermal strain field:

$$\underline{\underline{\varepsilon}}_{th} = \alpha(T - T_0)\underline{\underline{I}} = \alpha(\underline{X}) \left(T(\underline{X}) - T_0(\underline{X})\right)\underline{\underline{I}},$$

  where α is the coefficient of thermal expansion (CTE), T and T_0 are the current and reference temperature fields, respectively.

- The stress is defined by the elastic strain:

$$\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}_{th})$$

Stress: 2D isotropic elasticity

- Remind isotropic stress/strain relationship:

$$\underline{\underline{\sigma}} = \frac{\nu E}{(1+\nu)(1-2\nu)} \mathrm{tr}(\underline{\underline{\varepsilon}}) \underline{\underline{\boldsymbol{I}}} + \frac{E}{1+\nu} \underline{\underline{\varepsilon}}$$

- Stress (in Voigt notations): $\underline{\underline{\sigma}} \quad \Rightarrow \quad [S] = [\sigma_{xx}, \quad \sigma_{yy}, \quad \sigma_{xy}]^\top$
- In plane stress $\sigma_{zz} = 0$, $\varepsilon_{zz} = \frac{\nu}{\nu - 1}(\varepsilon_{xx} + \varepsilon_{yy})$
- In plain strain $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$, $\varepsilon_{zz} = 0$
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- Matrix $[\mathbf{D}]$ in plane strain $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$:

$$[\mathbf{D}] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2^{*} \end{bmatrix}$$

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$$[\mathbf{D}] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2^* \end{bmatrix}$$

---

*Factor $1/2$ appears because $[E]$ contains γ_{xy} and not ε_{xy}.

Stress: general case

Voigt notations in 3D case

- Stress tensor: $\underline{\underline{\sigma}} \rightarrow [S] = [\sigma_{xx},\ \sigma_{yy},\ \sigma_{zz},\ \sigma_{xy},\ \sigma_{yz},\ \sigma_{xz}]^\top$
- Strain tensor: $\underline{\underline{\varepsilon}} \rightarrow [E] = [\varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \gamma_{xy},\ \gamma_{yz},\ \gamma_{xz}]^\top$
- Hooke's law: $[S] = [\mathbf{D}]\,[E]$

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- Hooke's law: $[S] = [\mathbf{D}]\,[E]$
- Isotropic elasticity (two constants E, ν):

$$[\mathbf{D}] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

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- Cubic elasticity (3 constants E, ν, μ), in material axes, takes the form:

$$[\mathbf{D}] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}$$

Stress: general case II

Voigt notations in 3D case

- Transversely isotropic elasticity (5 constants $E_1, E_2, \nu_1, \nu_2, \mu_1$), in material axes:

$$[\mathbf{D}]_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 \end{bmatrix}$$

Stress: general case II

Voigt notations in 3D case

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- Orthotropic elasticity (9 constants $E_{xx}, E_{yy}, E_{zz}, \nu_{xy}, \nu_{yz}, \nu_{xz}, \mu_{xy}, \mu_{yz}, \mu_{xz}$), in material axes:

$$[\mathbf{D}]_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

Stress and reactions: element's equilibrium II

- According to the principle of virtual work:

$$\int_{\Omega} \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}} \, dV - \int_{\Omega} \underline{f}_V \cdot \delta \underline{u} \, dV = \int_{\Gamma_f} \underline{t}_0 \cdot \delta \underline{u} \, dS$$

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- Elastic stress $\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}_{th}) \Rightarrow [S] = [\mathbf{D}]\,([E] - [E_{th}])$
- Strain $\underline{\underline{\varepsilon}} \sim [E] = [B]^\top [u]$
- Volumetric force density $\underline{f}_v \sim [f_v] = [f_v^x, f_v^y, f_v^z]^\top$
- Virtual displacement $\delta \underline{u} \sim [N]^\top \delta [u]$

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- The discretized form of the virtual work:

$$\int_{\Omega^h} \left\{ ([\mathbf{D}]\,([E] - [E_{th}]))^\top \delta[E] - [f_v]^\top [N_i]^\top \delta[\boldsymbol{u}] \right\} dV = \int_{\Gamma_f^h} \underline{t}_0(\underline{X})[N_i]^\top \, dS \; \delta[\boldsymbol{u}]$$

Stress and reactions: element's equilibrium II

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$$[\boldsymbol{u}] \left[\int_{\Omega^h} [B]\,[\mathrm{D}]\,[B]^\top\, dV\right] \delta[\boldsymbol{u}] - \left[\int_{\Omega^h} \left([f_v]^\top [N_i]^\top + [E_{th}]^\top\, [\mathrm{D}]\,[B]^\top\right) dV\right] \delta[\boldsymbol{u}] = [f]^\top \delta[\boldsymbol{u}]$$

Stress and reactions: element's equilibrium II

- Balance of virtual work for a single element:

$$[\boldsymbol{u}]\left[\int\limits_{\Omega^h}[B]\,[\mathbf{D}]\,[B]^\top\,dV\right]\delta[\boldsymbol{u}] - \left[\int\limits_{\Omega^h}\left([f_v]^\top[N_i]^\top + [E_{th}]^\top\,[\mathbf{D}]\,[B]^\top\right)dV\right]\delta[\boldsymbol{u}] = [f]^\top\delta[\boldsymbol{u}]$$

- For arbitrary virtual displacements $\delta[\boldsymbol{u}]$:

$$\underbrace{\left[\int\limits_{V^e}[B]^\top\,[\mathbf{D}]\,[B]\,dV\right]}_{[\mathbf{K}^\mathbf{e}]}[\boldsymbol{u}] + \underbrace{\left[\int\limits_{V^e}\left(-[f_v]^\top[N_i]^\top - [B]\,[\mathbf{D}]\,[E_{th}]\right)dV\right]}_{[f^e_{int}]} = \underbrace{[f]}_{[f^e_{ext}]}$$

- System of equations linking displacements and reactions:

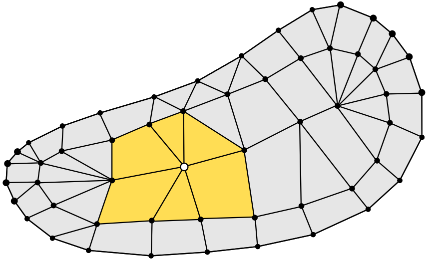
$$\boxed{[\mathbf{K}^\mathbf{e}]\,[u^e] + [f^e_{int}] = [f^e_{ext}]}$$

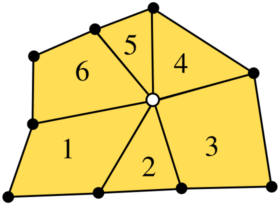
Assembly

- At every internal node the total force should be zero:

$$\sum_e [f^e_{ext}] = 0$$

summation over all elements e attached to this node.





- Summation over all nodes gives:

$$[\mathbf{K}]\,[u] + [f_{int}] = 0$$

Dirichlet boundary conditions

- From the weak form:

$$\int_{\Omega} \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}\, dV - \int_{\Omega} \underline{f}_V \cdot \delta\underline{u}\, dV = \int_{\Gamma_f} \underline{t}_0 \cdot \delta\underline{u}\, dS, \quad \text{for } \underline{u} \in \mathbb{U} = \{\underline{u} \in H^1(\Omega) | \underline{u} = \underline{u}_0 \text{ on } \Gamma_u\}$$

- Alternative (approximate) formulation with penalty (penalty factor ϵ):

$$\int_{\Omega} \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}\, dV - \int_{\Omega} \underline{f}_V \cdot \delta\underline{u}\, dV = \int_{\Gamma_f} \underline{t}_0 \cdot \delta\underline{u}\, dS + \int_{\Gamma_u} \epsilon(\underline{u}_0 - \underline{u}) \cdot \delta\underline{u}\, dS,$$

for $\underline{u} \in \mathbb{U} = \{\underline{u} \in H^1(\Omega)\}$

- Then

$$a(\underline{u}, \delta\underline{u}) = \int_{\Omega} \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}\, dV + \int_{\Gamma_u} \epsilon\underline{u} \cdot \delta\underline{u}\, dS$$

$$L(\delta\underline{u}) = \int_{\Omega} \underline{f}_V \cdot \delta\underline{u}\, dV + \int_{\Gamma_f} \underline{t}_0 \cdot \delta\underline{u}\, dS + \int_{\Gamma_u} \epsilon\underline{u}_0 \cdot \delta\underline{u}\, dS$$

Dirichlet boundary conditions

Dirichlet BC

- Use penalty method to enforce prescribed displacements: array $[\boldsymbol{u_0}] = [0 \ldots 0 \; u_{i0} \; 0 \ldots 0 \; u_{j0} \; 0]$
- Diagonal selection matrix $[\mathbf{I^s}]$ with ones at prescribed degrees of freedom (DOFs):

$$[\mathbf{I^s}] = \begin{bmatrix} 0 & \ldots & 0 & \overbrace{0}^{i} & 0 & \ldots & 0 & \overbrace{0}^{j} & 0 & \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \} \; i \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \} \; j \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \end{bmatrix}$$

- Then the system is changed to

$$([\mathbf{K}] + \epsilon\,[\mathbf{I^s}])\,[\boldsymbol{u}] = [\boldsymbol{f_{ext}}] - [\boldsymbol{f_{int}}] + \epsilon[\boldsymbol{u_0}]$$

where ϵ is the penalty coefficient such that $\epsilon \gg \max(K_{ij})$, and $[\mathbf{I}]$ is the identity matrix.

- Alternatively, (i) a direct DOF elimination or (ii) Lagrange multipliers could be used.

Neumann boundary conditions

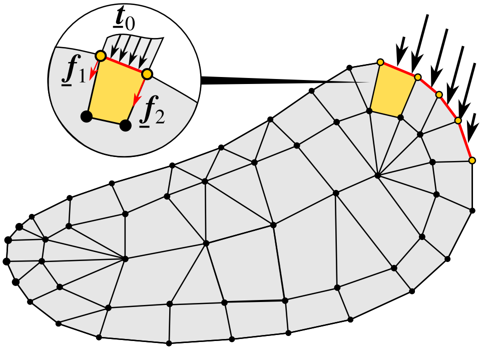
Neumann BC

- Surface traction $\underline{t}_0$ at Γ_f
- Virtual work of surface traction over one element:

$$\int_{\Gamma_f^e} \underline{t}_0 \cdot \delta \underline{u} \, d\Gamma = \underline{f}^i_{ext} \cdot \delta \underline{u}^e_i$$

- Then

$$[\boldsymbol{f}^{\boldsymbol{i}}_{\boldsymbol{ext}}] = \int_{\Gamma_f^e} [\boldsymbol{t}_0][N]^\top \, d\Gamma$$



Discrete system of equations

- Balance of virtual work for the whole body:

$$\underbrace{\left[\int_V [B]^\top [\mathbf{D}]\,[B]\,dV\right]}_{[\mathbf{K}]}[u] = \underbrace{\int_{\Gamma_f} [t_0]^\top [N]^\top\, d\Gamma}_{[f_{ext}]} + \underbrace{\left[\int_V \left([f_v]^\top [N_i]^\top + [B]\,[\mathbf{D}]\,[E_{th}]\right) dV\right]}_{-[f_{int}]}$$

- System of equations linking displacements and reactions:

$$\boxed{[\mathbf{K}]\,[u] = [f_{ext}] - [f_{int}]}$$

- Stiffness matrix $[\mathbf{K}]$
- Vector of degrees of freedom (DOFs) $[u]$
- Right hand term (vector of forces) $[f_{ext}] - [f_{int}]$

Evaluation of the integrals

- Weak form (recall):

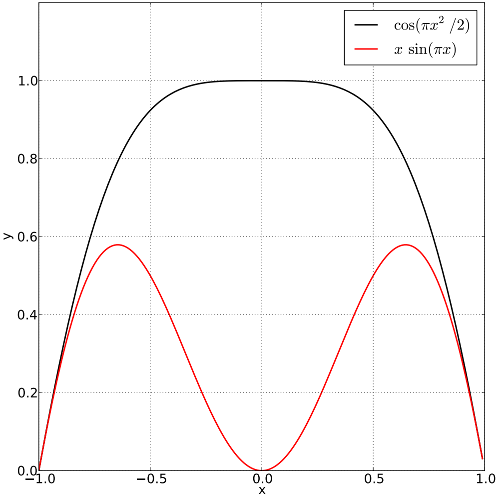
$$\underbrace{\left[\int_V [B]^\top [\mathbf{D}] [B]\, dV\right]}_{[\mathbf{K}]} [u] = \underbrace{\int_{\Gamma_f} [t_0]^\top [N]^\top \, d\Gamma}_{[f_{ext}]} + \underbrace{\left[\int_V \left([f_v]^\top [N_i]^\top + [B][\mathbf{D}][E_{th}]\right) dV\right]}_{-[f_{int}]}$$

- Exact integration: $\int_a^b f(x)dx = F(b) - F(a)$ (not always possible)
- Approximate integration (trapezoidal rule, Simpson's rule)
- Gauss quadrature: $\int_a^b f(x)dx \approx \sum_{i=1}^{N_{GP}} w_i f(x_i)$
- Gauss points x_i and weights w_i with $i = 1, N_{GP}$
- Integration is exact for polynomials of order $2N_{GP} - 1$
- Tabulated data for x_i, w_i (1D, 2D, 3D integration)

Evaluation of the integrals: example

- Function $f(x) = \cos(\pi x^2/2)$
  - $N_{GP} = 1$: error ≈ 28.22 %
  - $N_{GP} = 2$: error ≈ 11.04 %
  - $N_{GP} = 3$: error ≈ 1.14 %
  - $N_{GP} = 4$: error ≈ 0.14 %
  - $N_{GP} = 5$: error ≈ 0.01 %

- Function $f(x) = x\ \sin(\pi x)$
  - $N_{GP} = 1$: error ≈ 100.00 %
  - $N_{GP} = 2$: error ≈ 76.05 %
  - $N_{GP} = 3$: error ≈ 12.07 %
  - $N_{GP} = 4$: error ≈ 0.80 %
  - $N_{GP} = 5$: error ≈ 0.03 %



Evaluation of the integrals II

- Consider: $\int_V [\boldsymbol{B}]^\top [\mathbf{D}] [\boldsymbol{B}] \, dV = \sum_{e=1}^{N_e} \int_{V_e} [\boldsymbol{B}]^\top [\mathbf{D}] [\boldsymbol{B}] \, dV$
- Transpose to the parametric space or mapping (in 2D case):

$$\int_{V_e} [\boldsymbol{B}(\xi,\eta)]^\top [\mathbf{D}] [\boldsymbol{B}(\xi,\eta)] \, dV = \int_{-1}^{1} \int_{-1}^{1} [\boldsymbol{B}(\xi,\eta)]^\top [\mathbf{D}] [\boldsymbol{B}(\xi,\eta)] \det([\mathbf{J}]) \, d\xi d\eta$$

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- Finally:

$$[\mathbf{K}] = \int\limits_V [\boldsymbol{B}]^\top [\mathbf{D}] [\boldsymbol{B}] \, dV \approx \sum_{e=1}^{N_e} \sum_{GP=1}^{N_{GP}} [\boldsymbol{B}^e(\xi_{GP},\eta_{GP})]^\top [\mathbf{D}] [\boldsymbol{B}^e(\xi_{GP},\eta_{GP})] \det([\,J^e(\xi_{GP},\eta_{GP})\,]) w_{GP}$$

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Evaluation of the integrals III

- If $N(\xi, \eta) = P_p$ is a polynomial of order p, then $[\,\mathbf{J}\,] = P_{(p-1)}$, $[\boldsymbol{B}] = \frac{P_{2(p-1)}}{Q_{(p-1)}}$
- **Remark I:** Gauss quadrature is exact for $p = 1$ and approximate if $p > 1$.
- **Remark II:** Stress and strains are exactly evaluated only in Gauss points, in all other points they are extrapolated/interpolated
- **Remark III:** Underintegration may lead to zero-energy deformation modes (which have to be stabilized in FE software)

Evaluation of the integrals: quadrilateral 2D element

- Shape functions:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

- Shape function derivatives:

$$N_{1,\xi} = -\frac{1}{4}(1-\eta), \quad N_{2,\xi} = \frac{1}{4}(1-\eta)$$
$$N_{3,\xi} = \frac{1}{4}(1+\eta), \quad N_{4,\xi} = -\frac{1}{4}(1+\eta)$$

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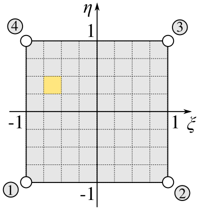
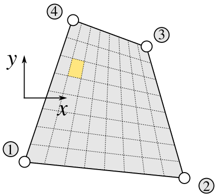
- Determinant of Jacobian ($dA = \det[\,\mathbf{J}\,]d\xi d\eta$):

$$\det([\,\mathbf{J}\,]) =$$
$$\frac{1}{16}\Big[\big((1-\eta)(x_2-x_1)+(1+\eta)(x_3-x_4)\big)\big((1+\xi)(y_3-y_2)+(1-\xi)(y_4-y_1)\big)-$$
$$-\,((1-\eta)(y_2-y_1)+(1+\eta)(y_3-y_4))((1+\xi)(x_3-x_2)+(1-\xi)(x_4-x_1))\Big]$$

Parametric space



Physical space



Evaluation of the integrals: quadrilateral 2D element

- Shape functions:

  $N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$

  $N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

- Shape function derivatives:

  $N_{1,\xi} = -\frac{1}{4}(1-\eta), \quad N_{2,\xi} = \frac{1}{4}(1-\eta)$

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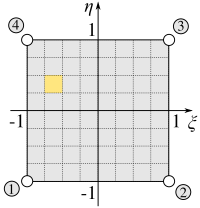
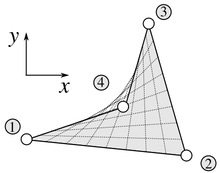
  $-\ ((1-\eta)(y_2-y_1)+(1+\eta)(y_3-y_4))((1+\xi)(x_3-x_2)+(1-\xi)(x_4-x_1))\Big]$

- **Warning:** to ensure $\det([\mathbf{J}]) > 0$ the element should remain convex

Parametric space



Physical space



Solvers

Problem: Find $[u]$ such that $\boxed{[\mathbf{K}]\,[u] = [f]}$, i.e. $[u] = [\mathbf{K}]^{-1}\,[f]$

- **Iterative solvers**
  The solution is approached iteratively, does not require much memory, restrictions to matrix type, sensitive to matrix conditioning, a preconditioner is often needed.
  - Gauss-Seidel method (GS)
  - Conjugate gradient method (CG)
  - Generalized minimum residual method (GMRES)
  - ...
- **Direct solvers**
  The solution is provided directly, no restrictions on matrix type, less sensitive to matrix conditioning, based on LU *or Cholesky decomposition*
  - Frontal
  - Sparse direct
  - ...

Convergence

Mesh and interpolation order convergence

- For Sobolev spaces[1] $\underline{\boldsymbol{u}} \in \mathbb{W}^{s,p}$, $s, p \in \mathbb{N}$ and their norm: $\|\underline{\boldsymbol{u}}\|_{\mathbb{W}^{s,p}} = \left[\int_\Omega \sum_{\alpha=0}^{s} \left(\frac{\partial^\alpha \underline{\boldsymbol{u}}}{\partial \underline{\boldsymbol{x}}^\alpha} \cdot \frac{\partial^\alpha \underline{\boldsymbol{u}}}{\partial \underline{\boldsymbol{x}}^\alpha} \right)^p dV \right]^{1/p}$
- For Sobolev space $\mathbb{H}^1$:

$$\|\underline{\boldsymbol{u}}\|_{\mathbb{H}^1} = \sqrt{\int_\Omega \left(\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{u}} + l^2 \nabla \underline{\boldsymbol{u}} : \nabla \underline{\boldsymbol{u}} \right) dV}$$

[1] The solution is usually sought in physically meaningful Sobolev space $W^{1,2}$, i.e. Sobolev space $\mathbb{H}^1$.

Mesh and interpolation order convergence

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- If there's no re-entrant corners and boundary conditions are "gentle", then displacements converge as :

$$\frac{||\underline{u} - \underline{u}^h||_{H^0}}{||\underline{u}||_{H^0}} \leq C_u h^{p+1}$$

where $\underline{u}$, $\underline{u}^h$ are the true and approximate solutions, p is the interpolation order of shape functions $N(\xi, \eta)$ and h is the element size.

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- And that stresses/strains converge as:

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Mesh and interpolation order convergence

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- If there's no re-entrant corners and boundary conditions are "gentle", then displacements converge as :

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  where $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{u}}^h$ are the true and approximate solutions, p is the interpolation order of shape functions $N(\xi, \eta)$ and h is the element size.
- And that stresses/strains converge as:

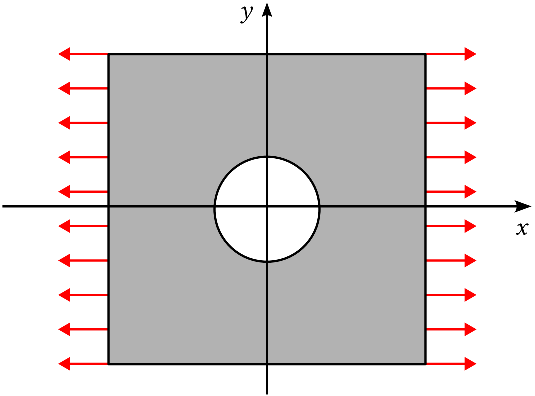
$$\frac{||\underline{\boldsymbol{u}} - \underline{\boldsymbol{u}}^h||_{H^1}}{||\underline{\boldsymbol{u}}||_{H^1}} \leq C_\sigma h^p$$

- Therefore, to obtain a converged solution we can either increase interpolation order p (**p-refinement**) or decrease h (**h-refinement**)

[1] The solution is usually sought in physically meaningful Sobolev space $W^{1,2}$, i.e. Sobolev space $\mathbb{H}^1$.

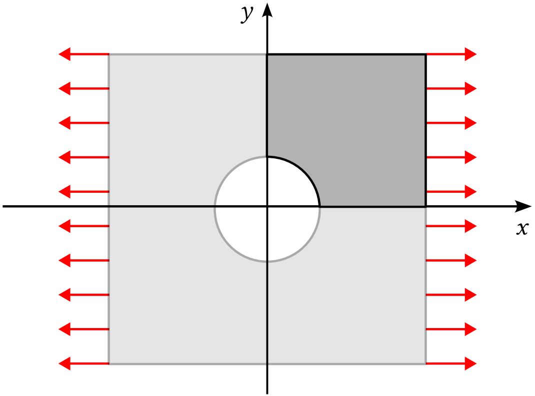
Example

Tension of a rectangular sheet with a hole



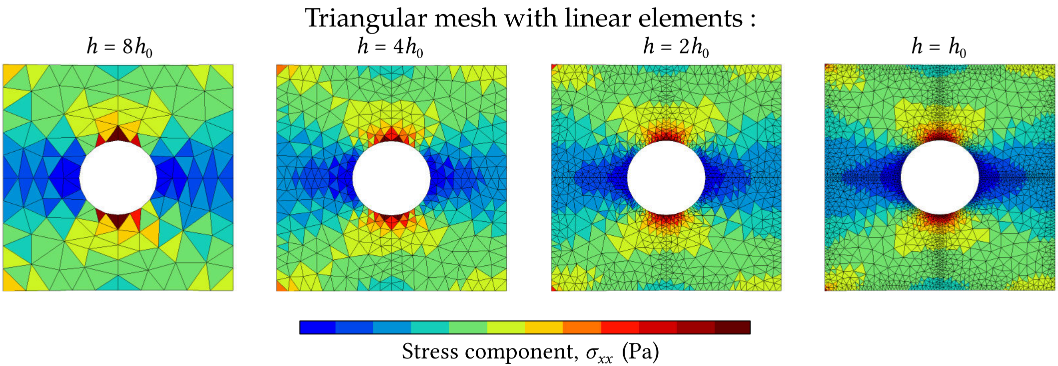
Example

Tension of a rectangular sheet with a hole



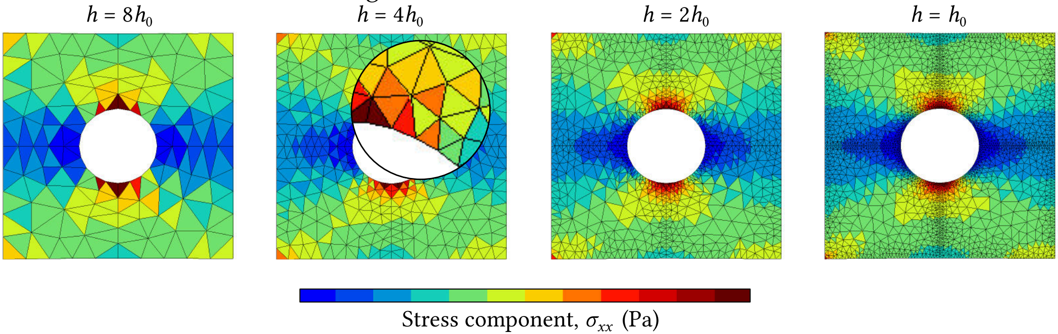
The symmetry is used to reduce the computational cost☺

Example

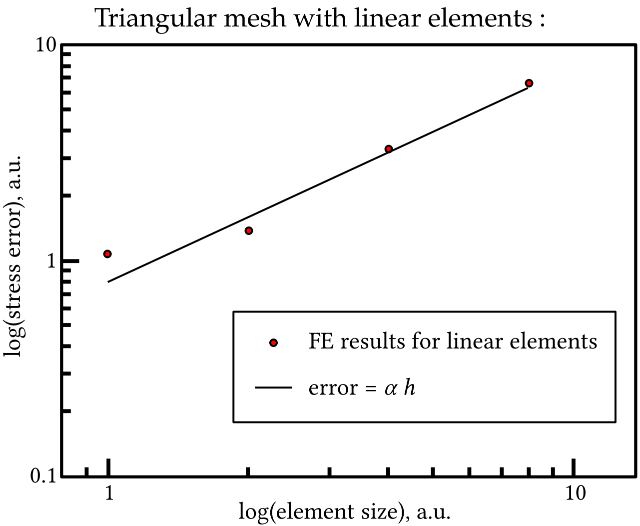

Triangular mesh with linear elements :
$h = 8h_0$
$h = 4h_0$
$h = 2h_0$
$h = h_0$
Stress component, σ_{xx} (Pa)

Example

Triangular mesh with linear elements :



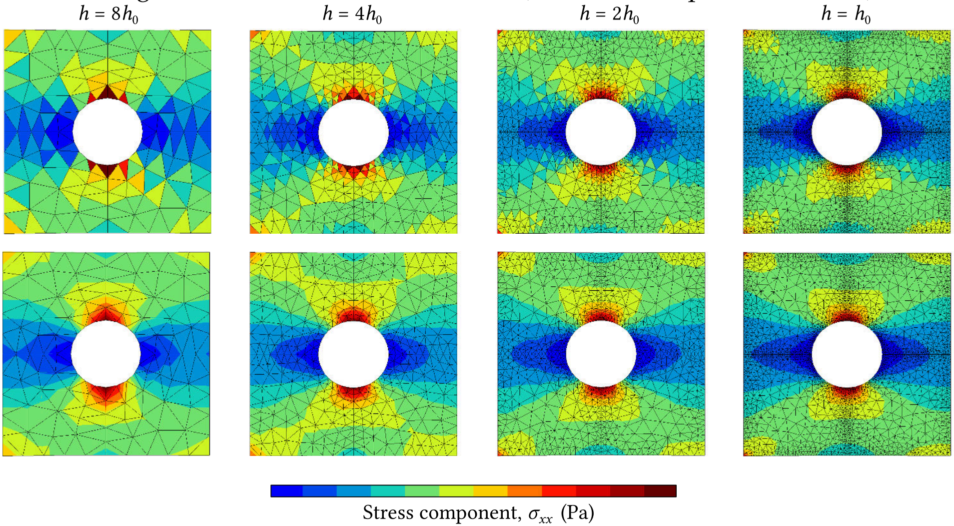
Example



Triangular mesh with linear elements :
10
1
0.1
1
10
log(stress error), a.u.
log(element size), a.u.
FE results for linear elements
error = α h

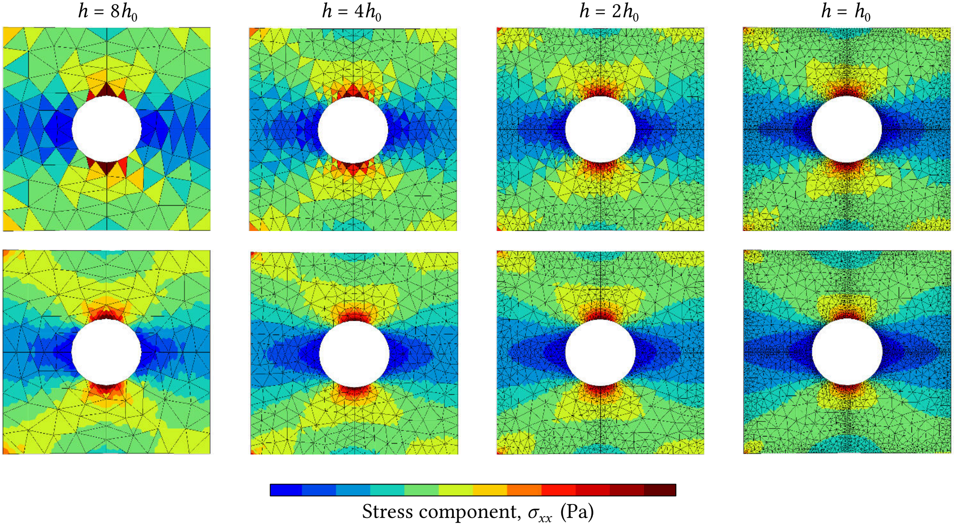
Example

Triangular mesh with linear elements (with contour plot stress field):



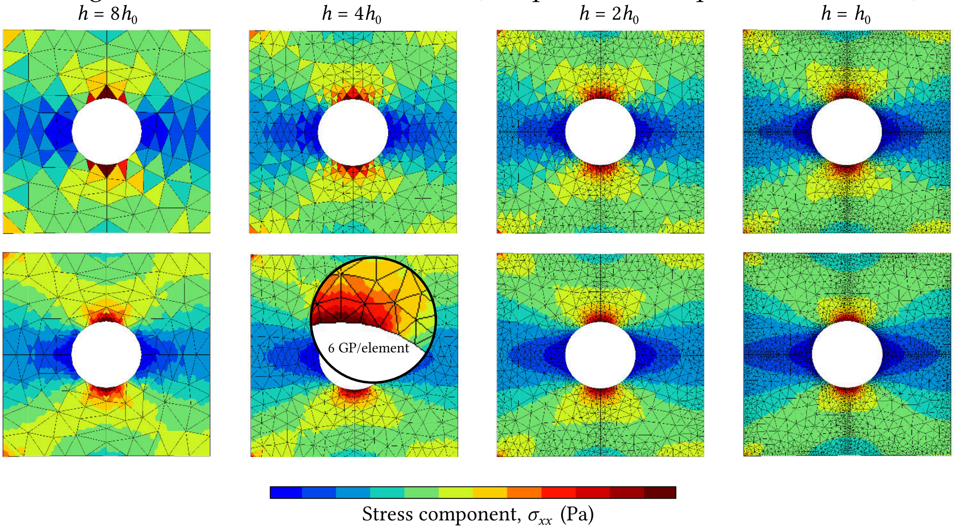
Example

Triangular mesh with linear elements (comparison with quadratic elements):



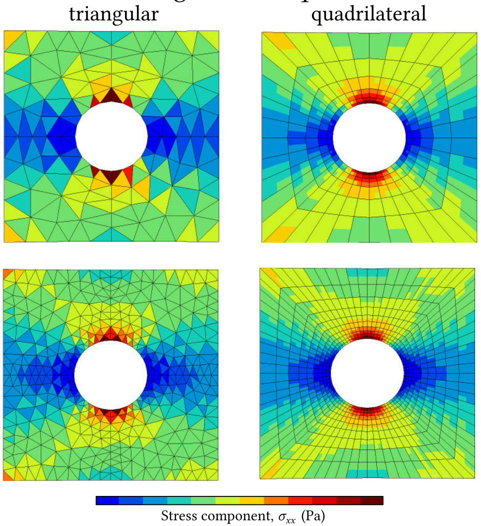
Example

Triangular mesh with linear elements (comparison with quadratic elements):



Example

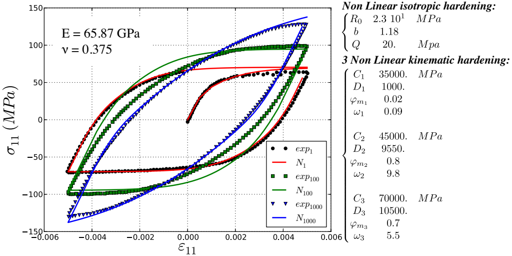
Comparison of triangular and quadrilateral meshes:

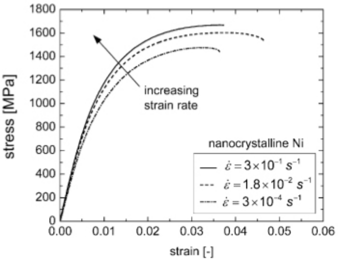


Nonlinear FEM

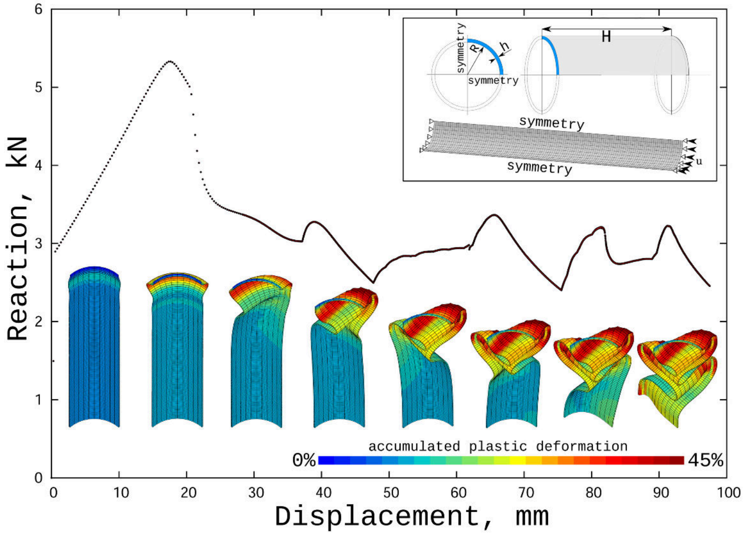
Types of nonlinearity

- Material behavior (viscoelasticity, plasticity, damage)
- Nonlinear geometry = finite deformations and/or rotations $\Omega(t) \neq \Omega(t_0)$, infinitesimal strain tensor $\underline{\underline{\varepsilon}}$ is not applicable
- Fracture (crack propagation: remeshing of X-FEM)
- Contact, friction, wear
- Coupled thermomecanical or fluid/solid problems

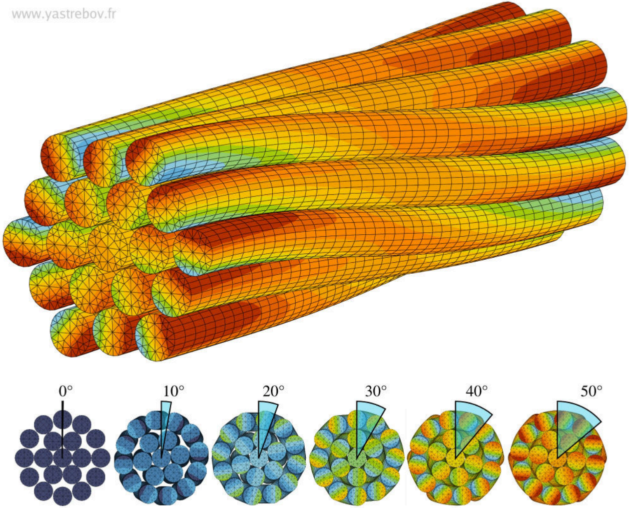




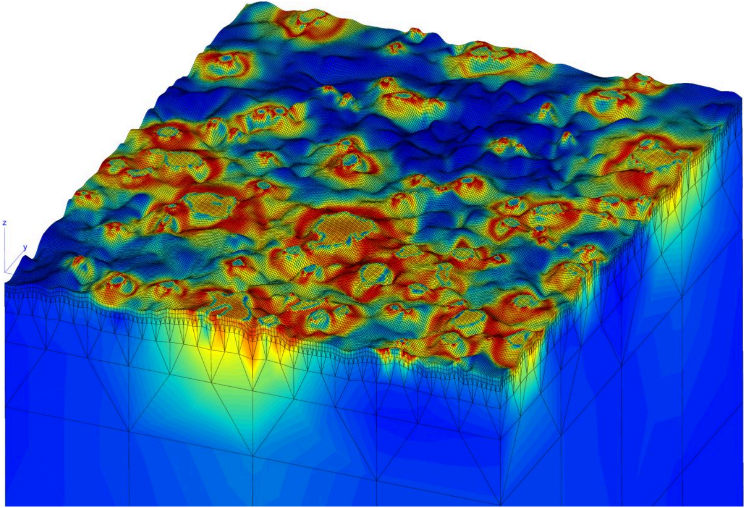
Post-buckling behavior with self-contact



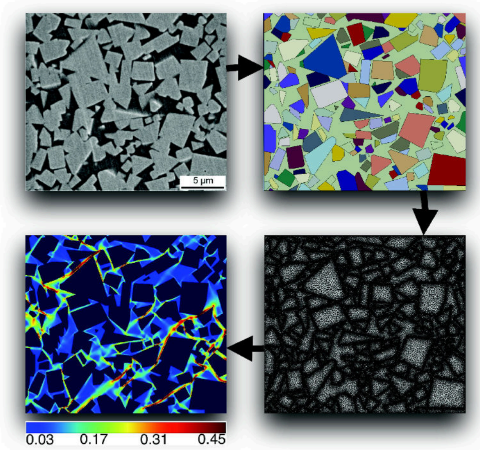
Twisting multi-strand wire

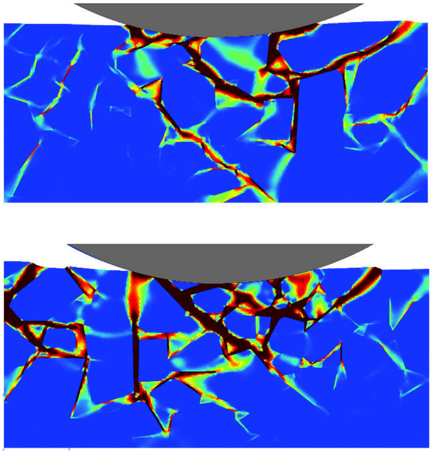


Contact of a rough surface

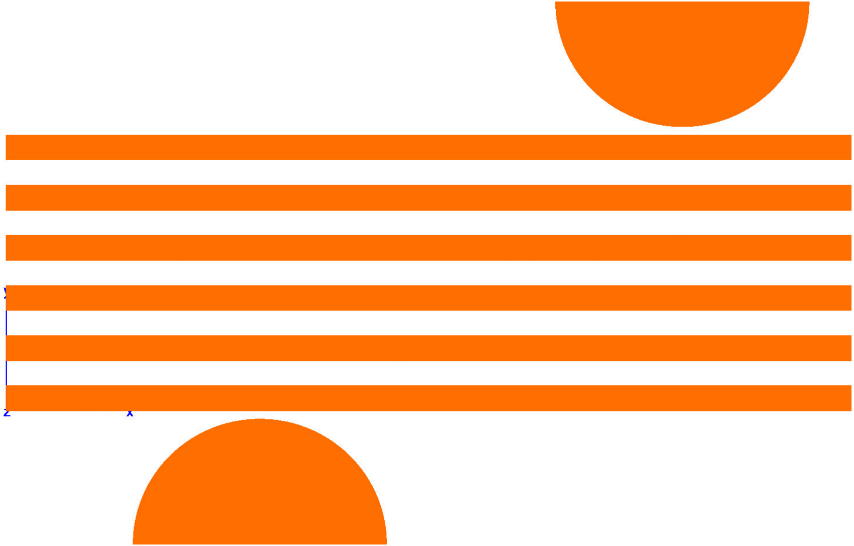


Impact of WC/Co composite





Multi-contact problem



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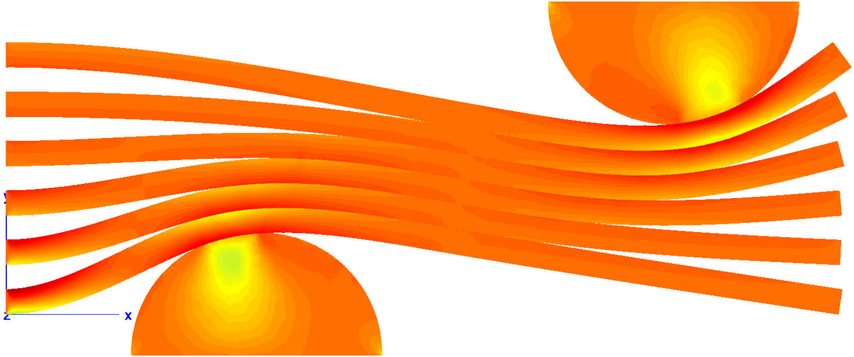

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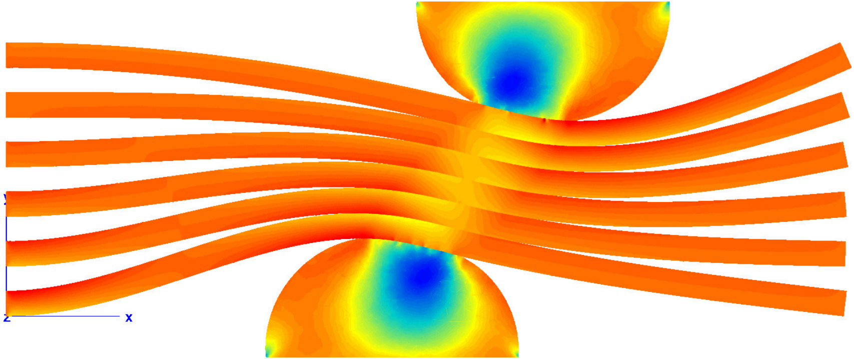

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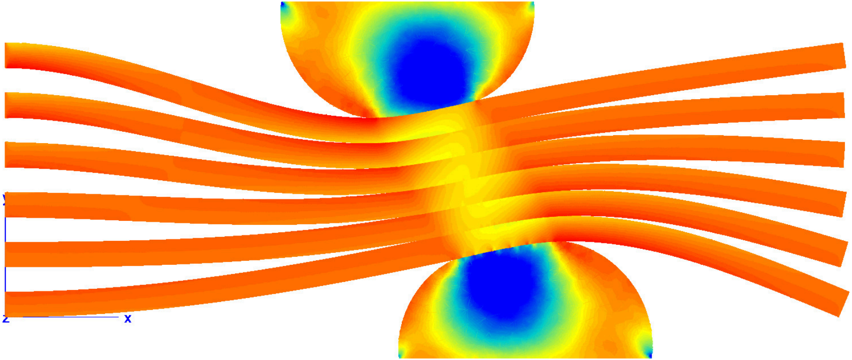
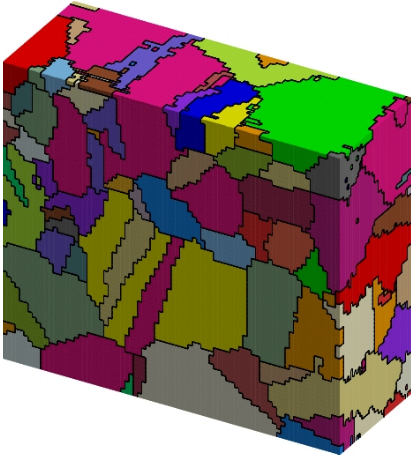
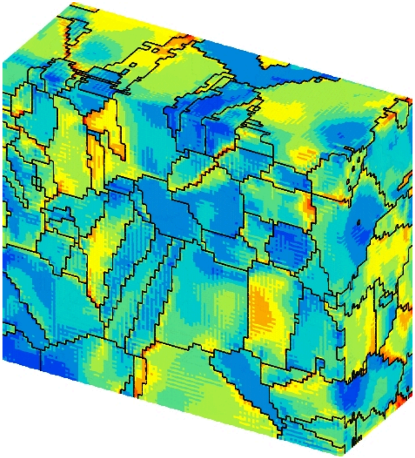
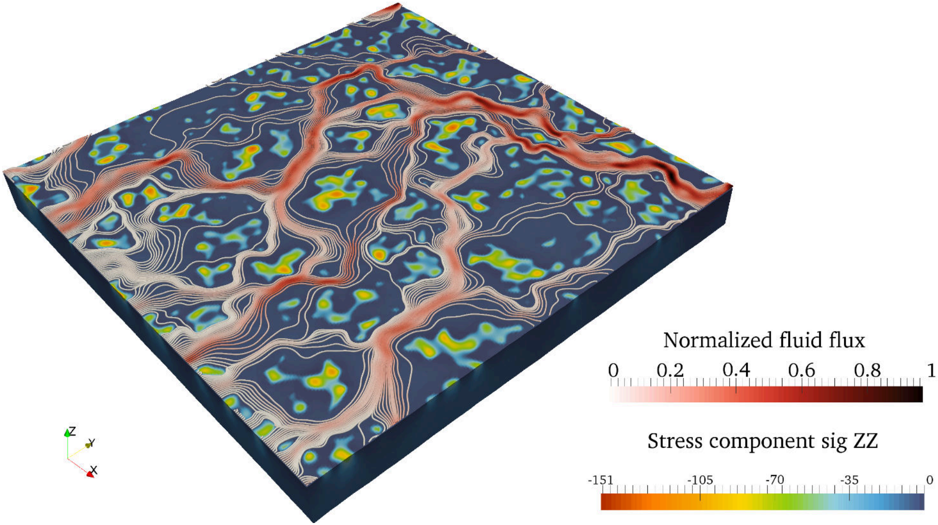


Polycristalline material





Coupled thin flow in contact interface



Conclusion

- The linear Finite Element Method is widely used in mechanical engineering
- To get to a matrix formulation (linear system of equations)

$$[K][u] = [f]$$

we need to compute:

  - a matrix $[B]$ at every Gauss point (GP)
  - a trivial matrix $[D]$ (which changes from GP to GP only if we have heterogeneous solid)
  - a vector of external forces $[f_{ext}]$ (Neumann boundary condition)
  - Dirichlet boundary conditions are imposed either using penalty method or matrix rearrangement
- The system is solved using your preferable solver
(see Christophe Bovet's (ONERA) lecture)

Recommended literature

FEM from mechanical engineering prospective

Merci de votre attention !