Optimization Under Constraints:
A Contact Problem

Vladislav A. Yastrebov

MINES Paris - PSL, CNRS
Centre des Matériaux, Evry, France

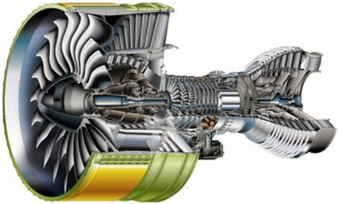
Semaine PSL @ MINES Paris
November 27, 2024

Outline

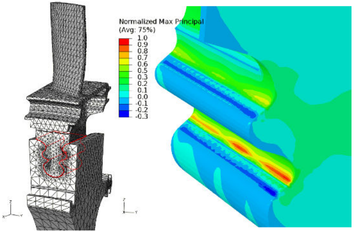
Introduction

Industrial and natural contact problems

1. Assembled parts, e.g. engines



Aircraft's engine GP 7200
www.safran-group.com

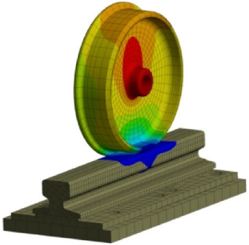


[1] M. W. R. Savage
J. Eng. Gas Turb. Power, 134:012501 (2012)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts

High speed train TGV `www.sncf.com`

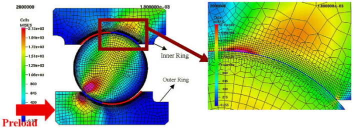


Wilde/ANSYS `wildeanalysis.co.uk`

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings

Bearings
www.skf.com



[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier
Mech. Syst. Signal Pr., 24:1068-1080 (2010)

Industrial and natural contact problems

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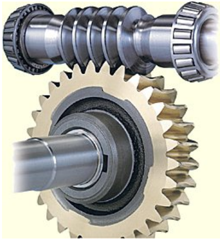
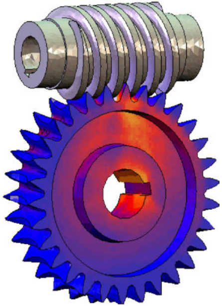


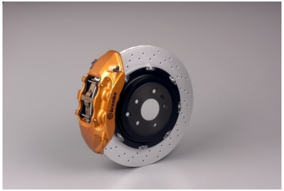
Helical gear www.tpg.com.tw



www.mscsoftware.com

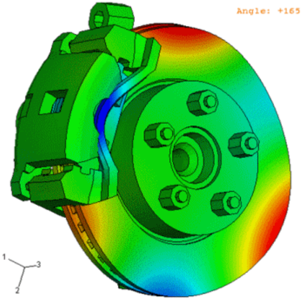
Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems



Assembled breaking system

www.brembo.com



www.mechanicalengineeringblog.com

Industrial and natural contact problems

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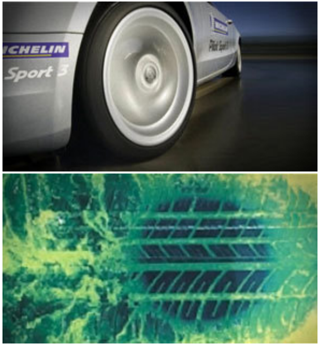
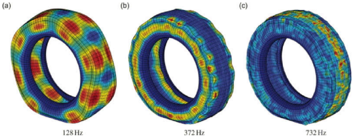
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact



Tire-road contact www.michelin.com

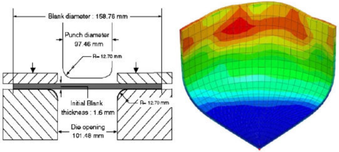


[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, J. Sound Vib., 309:20-39 (2008)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
2. Railroad contacts
3. Gears and bearings
4. Breaking systems
5. Tire-road contact
6. Metal forming

Deep drawing www.thomasnet.com



[1] G. Rousselier, F. Barlat, J. W. Yoon
Int. J. Plasticity, 25:2383-2409 (2009)

Industrial and natural contact problems

1. Assembled parts, e.g. engines
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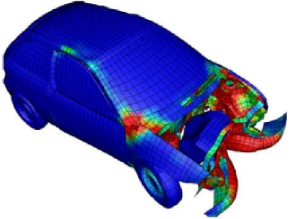
6. Metal forming
7. Crash tests

Crash-test www.porsche.com



[1] O. Klyavin, A. Michailov, A. Borovkov www.fea.ru

Industrial and natural contact problems

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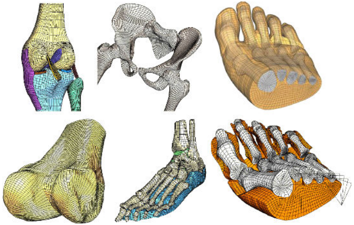
8. Biomechanics

Human articulations
www.sportssupplements.net



J. A. Weiss, University of Utah
Musculoskeletal Research Laboratories

Industrial and natural contact problems

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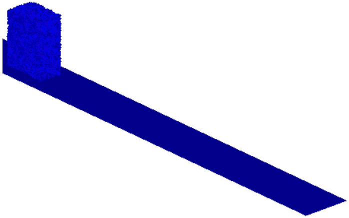
8. Biomechanics
9. Granular materials

Sand dunes www.en.wikipedia.org



E. Azema et al, LMGC90

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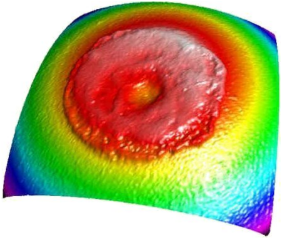
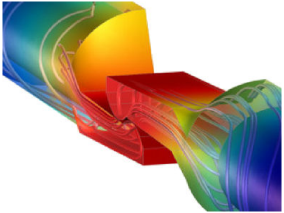
7. Crash tests
8. Biomechanics
9. Granular materials
10. Electric contacts



Damage at electric contact zone
www.taicaan.com



Simulation of electric current
www.comsol.com

Industrial and natural contact problems

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7. Crash tests
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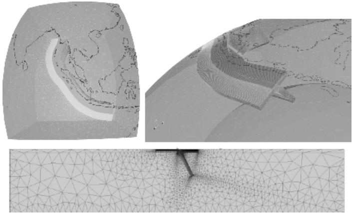
9. Granular materials
10. Electric contacts
11. Tectonic motions

San-Andreas fault, by M. Rightmire
www.sciencedude.ocregister.com



[1] J.D. Garaud, L. Fleitout, G. Cailletaud Colloque CSMA (2009)

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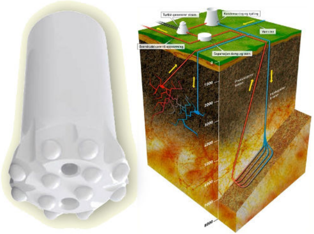
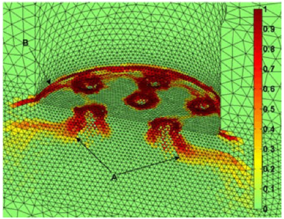
10. Electric contacts
11. Tectonic motions
12. Deep drilling



Drill Bit tool RobitRocktools*;*
*extraction of geothermal energy (*SINTEF, NTNU*)*



[1] T. Saksala, Int. J. Numer. Anal. Meth. Geomech. (2012)

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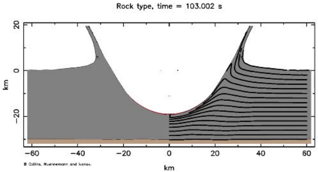
12. Deep drilling
13. Impact and fragmentation

Impact crater, Arizona
`www.MrEclipse.com` *et* `maps.google.com`



Simulation of formation of Copernicus crater
Yue Z., Johnson B. C., et al. Projectile remnants in central peaks of lunar impact craters. Nature Geo 6 (2013)

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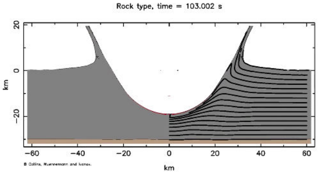
14. etc.

Impact crater, Arizona
www.MrEclipse.com *et* maps.google.com



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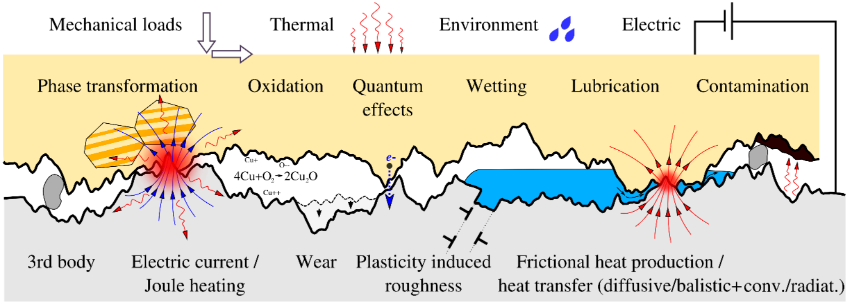
Physical and mathematical complexity

- Contact interface is hard to observe in situ
- Plenty of phenomena happen in the interface
- Strong thermo-mechanical or fluid-solid coupling in sliding
- Mathematical formulation is also non-trivial, hard to handle analytically
- **Robust and accurate computational framework is needed**



Interface between two solids in contact

Basics of Contact and Friction

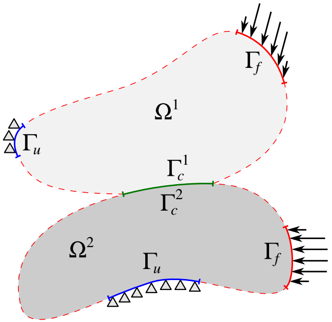
Equilibrium and contact conditions

- **Balance of momentum**

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0 & \text{on } \Gamma_f \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma_u \\ ? & \text{on } \Gamma_c \end{cases}$$

- **Frictionless contact conditions** *(intuitive)*

1. No penetration
2. No adhesion
3. No shear transfer



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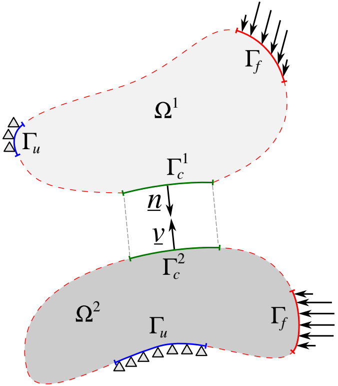

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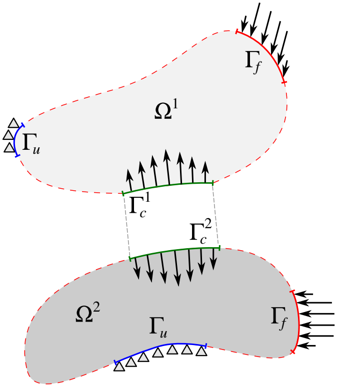

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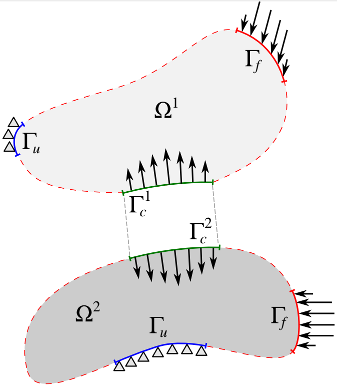

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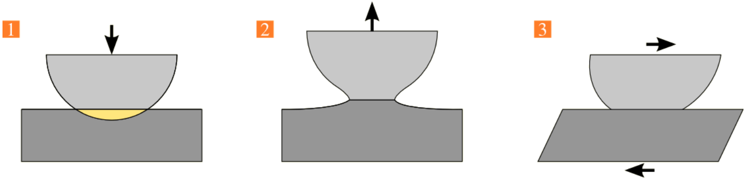

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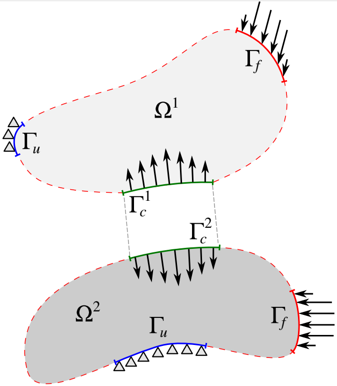

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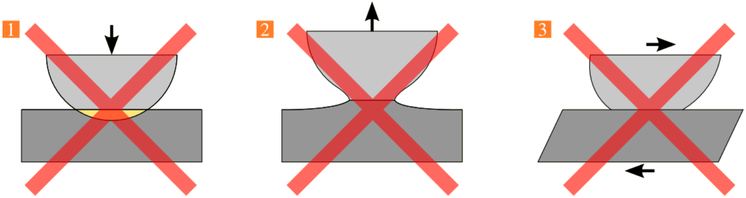
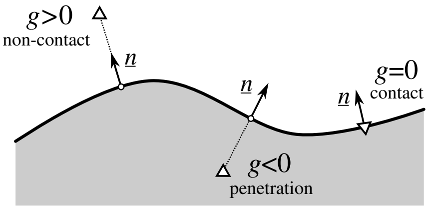


Gap function

- **Gap function** g
  - gap = – penetration
  - asymmetric function
  - defined for
    - separation $g > 0$
    - contact $g = 0$
    - penetration $g < 0$
  - governs normal contact
- **Master and slave split**

  Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

Gap function

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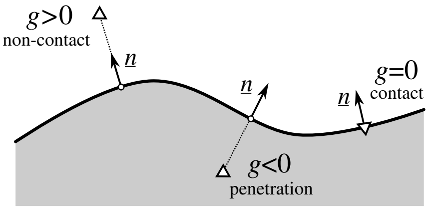
  Gap function is determined for all slave points with respect to the master surface
- **Normal gap**

$$g_n = \underline{n} \cdot \left[\underline{r}_s - \underline{\rho}(\xi_\pi) \right],$$

$\underline{n}$ is a unit normal vector, $\underline{r}_s$ slave point, $\underline{\rho}(\xi_\pi)$ projection point at master surface



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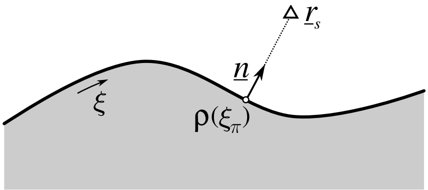


Definition of the normal gap

Consider existence and uniqueness

Frictionless or normal contact conditions

- **No penetration**
  Always non-negative gap

  $$g \geq 0$$

- **No adhesion**
  Always non-positive contact pressure

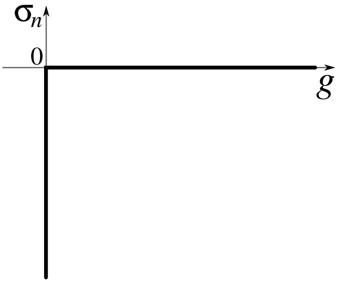
  $$\sigma_n^* \leq 0$$

- **Complementary condition**
  Either zero gap and non-zero pressure, or non-zero gap and zero pressure

  $$g \, \sigma_n = 0$$

- **No shear transfer** *(automatically)*

  $$\underline{\sigma}_t^{**} = 0$$



Scheme explaining normal contact conditions

$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$

$\underline{\sigma}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot \left(\underline{\underline{I}} - \underline{n} \otimes \underline{n}\right)$

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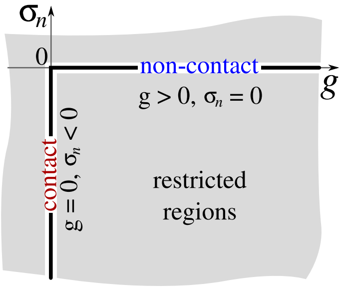


Improved scheme explaining normal contact conditions

Frictionless or normal contact conditions

In **mechanics:**

Normal contact conditions

≡

Frictionless contact conditions

≡

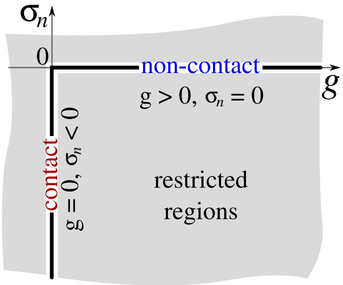
Hertz[1] *-Signorini*[2] *conditions*

≡

Hertz[1]*-Signorini*[2]*-Moreau*[3] *conditions*

also known in **optimization theory** as

Karush[4]*-Kuhn*[5]*-Tucker*[6] *conditions*



Improved scheme explaining normal contact conditions

$$g \geq 0, \qquad \sigma_n \leq 0, \qquad g\sigma_n = 0$$

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[1] **Heinrich Rudolf Hertz** (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

[2] **Antonio Signorini** (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

[3] **Jean Jacques Moreau** (1923-2014) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

[4] **William Karush** (1917–1997), [5] **Harold William Kuhn** (1925-2014) American mathematicians,

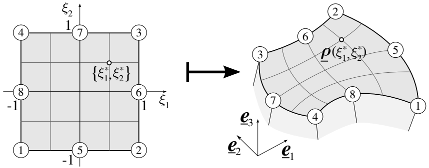
[6] **Albert William Tucker** (1905–1995) a Canadian mathematician.

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

$$\underline{\rho}(\xi_1,\xi_2,t)=\sum_{i=1}^{8}N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$



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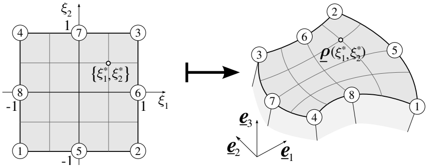
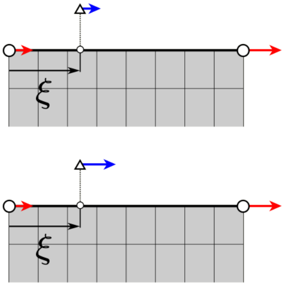
- **Tangential slip velocity $\underline{v}_t$** must take into account:
  - only tangential component
  - relative rigid body motion
  - master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.





Relative slip between a slave point and a deformable master surface

Relative sliding

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  - relative rigid body motion
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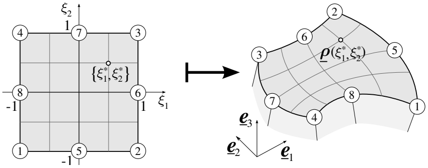
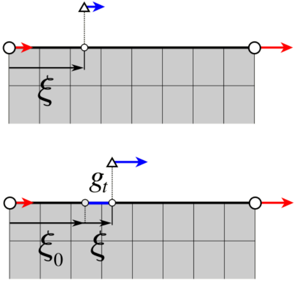
$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2}\dot{\xi}_2$$

where $\partial\underline{\rho}/\partial\xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.





Relative slip between a slave point and a deformable master surface

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1;1)$
- Mapping to real space

$$\underline{\rho}(\xi_1,\xi_2,t) = \sum_{i=1}^{8} N^i(\xi_1,\xi_2)\underline{\rho}^i(t)$$

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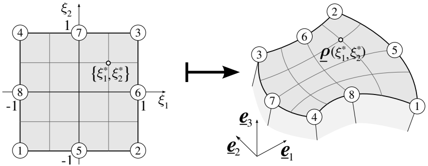

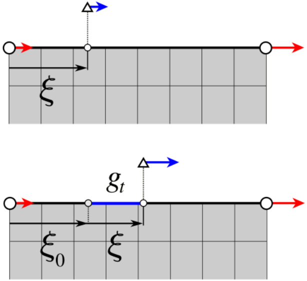

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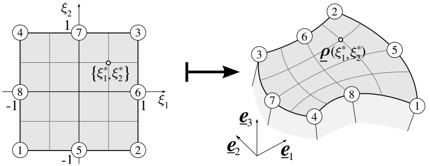

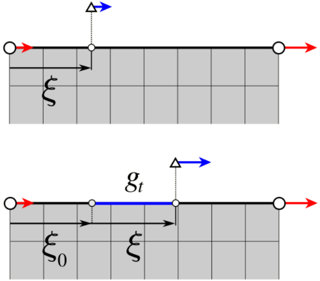

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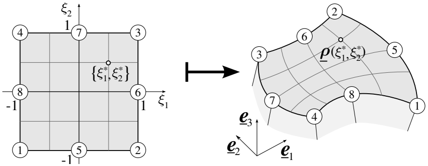

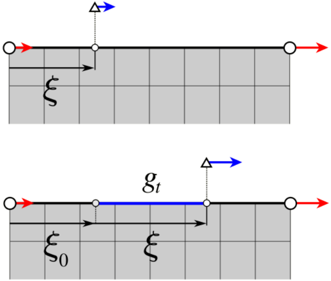

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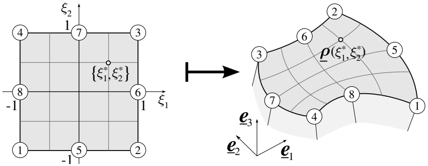
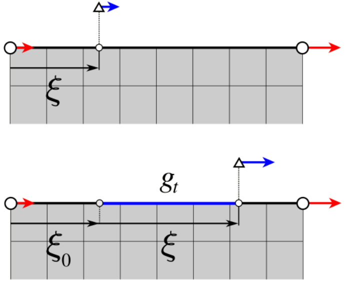




Relative slip between a slave point and a deformable master surface

Relative sliding: example

Consider a one-dimensional example:
P is a projection of A on segment BC.

$$x_P = \xi x_C + (1-\xi)x_B \qquad (1)$$

Velocity of the projection point

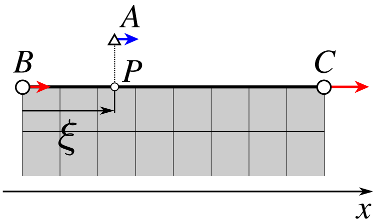
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1-\xi)\dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B)\dot{\xi}}_{\frac{\partial x_P}{\partial \xi}\dot{\xi}}$$

Substract the velocity of point x_P for fixed ξ

$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B)\dot{\xi} = \frac{\partial x}{\partial \xi}\dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left.\frac{\partial x}{\partial \xi}\right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

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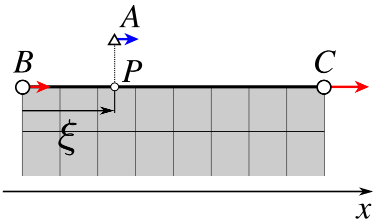
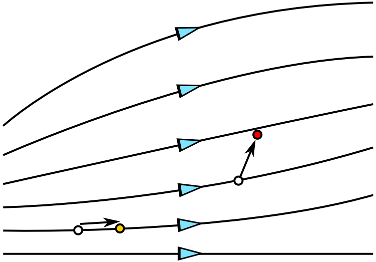


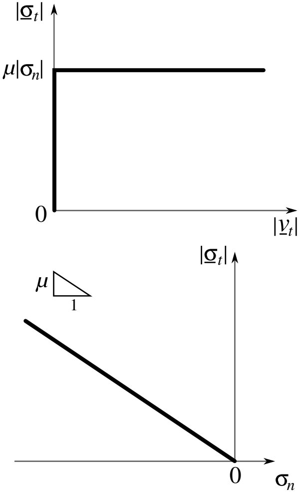
Example of a one-dimensional relative slip



Fisherman's analogy: observing sea flow around the boat.
Lie derivative: the change of a vector field along the change of another vector field

Amontons-Coulomb's friction

- **No contact** $g > 0,\ \sigma_n = 0$
- **Stick** $|\underline{v}_t| = 0$
  Inside slip surface/Coulomb's cone
  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$
- **Slip** $|\underline{v}_t| > 0$
  On slip surface/Coulomb's cone
  $$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$
- **Complementary condition**
  One is zero another one is not or vice versa
  $$|\underline{v}_t|\left(|\underline{\sigma}_t| - \mu|\sigma_n|\right) = 0$$
- **Direction of friction**
  Shear and sliding are collinear
  $$\underline{v}_t \parallel \underline{\sigma}_t$$



Scheme explaining frictional contact conditions

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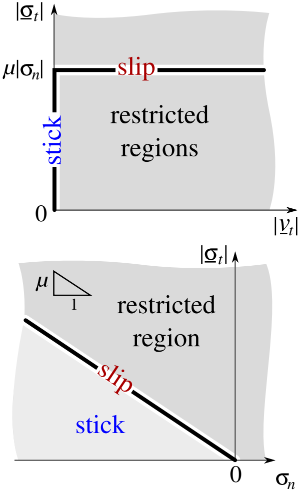

Improved scheme explaining frictional contact conditions

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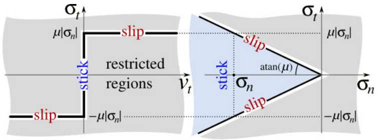
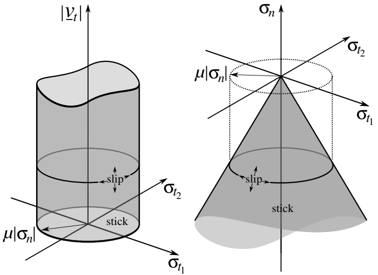


Scheme of 2D frictional contact

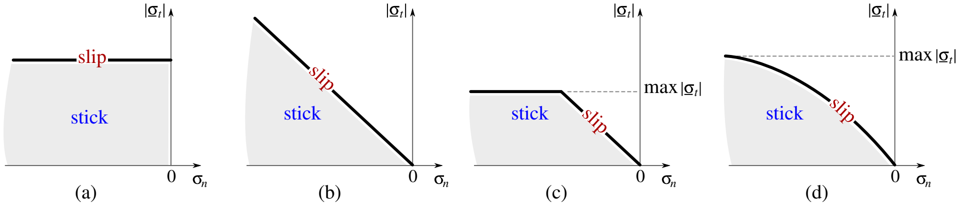


Scheme of 3D frictional contact

$$|\underline{v}_t| \geq 0, \quad |\underline{\sigma}_t| - \mu|\sigma_n| \leq 0, \quad |\underline{v}_t|\left(|\underline{\sigma}_t| - \mu|\sigma_n|\right) = 0, \quad \frac{\underline{\sigma}_t}{|\underline{\sigma}_t|} = -\frac{\underline{v}_t}{|\underline{v}_t|}$$

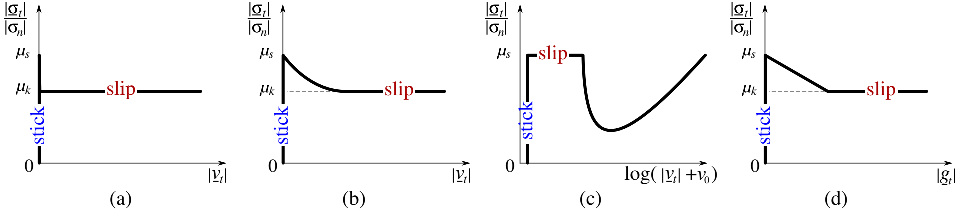
More friction laws

- **Static criteria**



(a) Tresca (b) Amontons-Coulomb (c) Coulomb-Orowan (d) Shaw

- **Kinetic criteria**



(a,b) velocity weakening (c) velocity weakening-strengthening (d) linear slip weakening

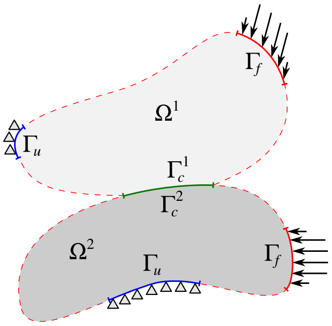
- μ_s static and μ_k kinetic coefficients of friction.

Towards a weak form

From strong to a weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$



Two solids in contact

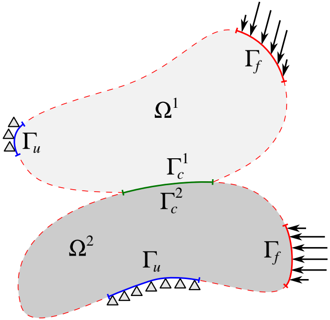
From strong to a weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \qquad + \text{B.C.}$$

- Balance of virtual works 

$$\boxed{\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta\underline{u} \, d\Gamma} + \int_{\Omega} \left[\underline{f}_v \cdot \delta\underline{u} - \underline{\underline{\sigma}} \cdot\cdot \, \delta\nabla\underline{u} \right] d\Omega = 0$$



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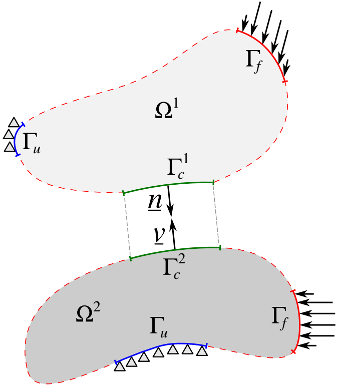
$$\boxed{\int_{\overline{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\rho} \, d\overline{\Gamma}_c^1 + \int_{\overline{\Gamma}_c^2} \underline{\nu} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{r} \, d\overline{\Gamma}_c^2} + \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma_f$$



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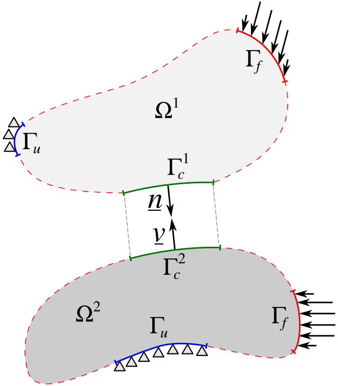

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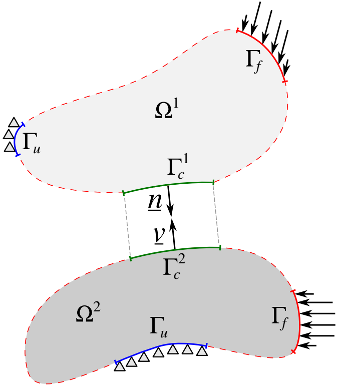
$$\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega + \underbrace{\boxed{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1}}_{\text{Contact term}} = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega$$



Two solids in contact

From strong to a weak form

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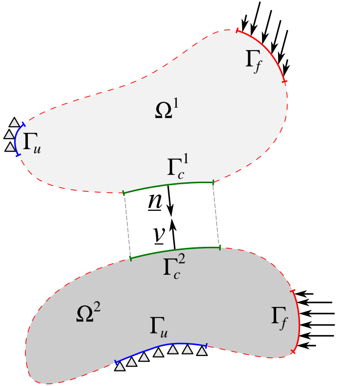
- Balance of virtual works 

$$\underbrace{\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta \nabla \underline{u} \, d\Omega}_{\text{Change of the internal energy}} + \underbrace{\boxed{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underset{\sim}{\sigma}{}_t^T \delta \underset{\sim}{\xi} \right) d\bar{\Gamma}_c^1}}_{\text{Contact term}} =$$

$$\underbrace{\int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega}_{\text{Virtual work of volume forces}}$$



Two solids in contact

- Functional **space**

$\delta \underline{u}, \underline{u} \in \mathbb{H}^1(\Omega)$ Sobolev space of the first order (function and its first derivate is square integrable) and $\underline{u}$ satisfy boundary conditions

From strong to a weak form

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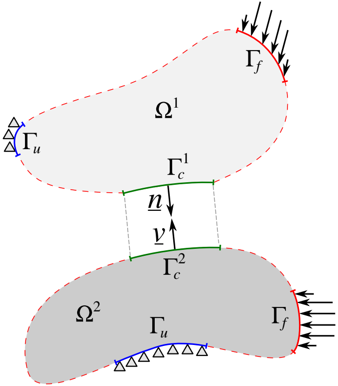

Two solids in contact

- Functional **subspace**

$\delta\underline{u}, \underline{u} \in \mathbb{H}^1(\Omega)$ Sobolev space of the first order (function and its first derivate is square integrable) and $\underline{u}$ satisfy boundary conditions and **contact conditions**, so we do optimization on a subset of $\mathbb{H}^1(\Omega)$.

Variational inequality

- Optimization problem for $F : \mathbb{V} \to \mathbb{R}$
- Find $u \in \mathbb{V}$ s.t. $\forall v \in \mathbb{V} : \; F(u) \leq F(v)$
- If $F \in C^1$ is convex then such minimizer u is a stationary point $F'|_u = 0$

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- If $\mathbb{K}$ is convex, then if $u \in \mathbb{K}$ is a minimizer, $\forall v \in \mathbb{K}, \theta \in [0, 1] : \quad F(u) \leq F(u + \theta(v - u))$

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- In the limit

$$\lim_{\theta \to 0} \frac{F(u + \theta(v-u)) - F(u)}{\theta} \geq 0$$

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$$\lim_{\theta \to 0} \frac{F(u + \theta(v - u)) - F(u)}{\theta} = F'(u)(v - u) \ge 0$$

Variational inequality

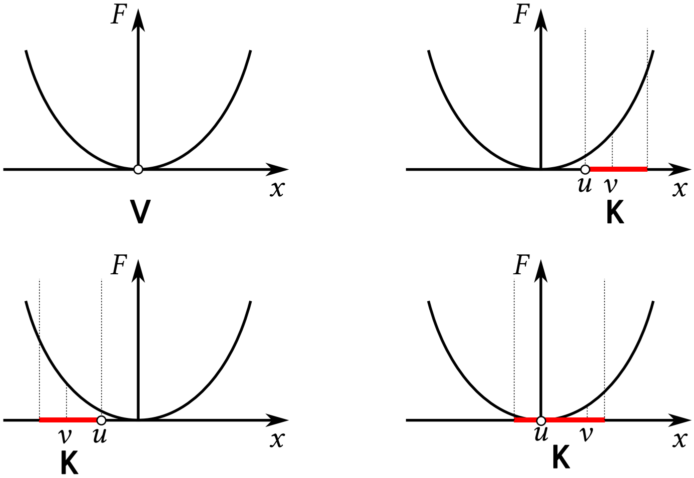
- Optimization problem for $F : \mathbb{V} \to \mathbb{R}$
- Find $u \in \mathbb{V}$ s.t. $\forall v \in \mathbb{V} : \; F(u) \le F(v)$
- If $F \in C^1$ is convex then such minimizer u is a stationary point $F'|_u = 0$
- However, finding minimizer of F on a subset $\mathbb{K} \subset \mathbb{V}$ changes the story
- If $\mathbb{K}$ is convex, then if $u \in \mathbb{K}$ is a minimizer, $\forall v \in \mathbb{K}, \theta \in [0,1] : \quad F(u) \le F(u + \theta(v - u))$
- In the limit

$$\lim_{\theta \to 0} \frac{F(u + \theta(v - u)) - F(u)}{\theta} = F'(u)(v - u) \ge 0$$

- Variational inequality for minimizer $u \in \mathbb{K} \subset \mathbb{V}$:

$$\boxed{F'(u)(v - u) \ge 0, \quad \forall v \in \mathbb{K}}$$

Example of variational inequality



Minimize $F(x)$ for $x \in \mathbb{K} \subset \mathbb{R}$, then the minimizer u satisfies

$$F'(u)(v-u) \geq 0, \quad \forall v \in \mathbb{K}$$

Variational inequality and a simplification

- Constrained minimization problem (variational inequality)[1,2]

$$\int_{\Omega} \underline{\underline{\boldsymbol{\sigma}}} \cdot\cdot \delta\nabla \underline{\boldsymbol{u}}\, d\Omega + \int_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi}\, d\bar{\Gamma}_c^1 \ \color{red}{\geq}\ \int_{\Gamma_f} \underline{\boldsymbol{\sigma}}_0 \cdot \delta \underline{\boldsymbol{u}}\, d\Gamma + \int_{\Omega} \underline{\boldsymbol{f}}_v \cdot \delta \underline{\boldsymbol{u}}\, d\Omega, \quad \underline{\boldsymbol{u}} \in \mathbb{L}, \delta\underline{\boldsymbol{u}} \in \mathbb{K}$$

$$\mathbb{L} = \left\{ \underline{\boldsymbol{u}} \in \mathbb{H}^1(\Omega) \;\middle|\; \underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}_0 \ \text{ on } \Gamma_u,\ g_n(\underline{\boldsymbol{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$
$$\mathbb{K} = \left\{ \delta\underline{\boldsymbol{u}} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{\boldsymbol{u}} = 0 \ \text{ on } \Gamma_u,\ g_n(\delta\underline{\boldsymbol{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

[1] Duvaut, G. and Lions, J.L., 1972. *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972
[2] Duvant, G. and Lions, J.L., 1976. *Inequalities in mechanics and physics*, Springer

Variational inequality and a simplification

- Constrained minimization problem (variational inequality)[1,2]

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int_{\bar{\Gamma}_c^1} \underset{\sim}{\sigma}_t^T \delta \underset{\sim}{\xi}\, d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega, \quad \underline{u} \in \mathbb{L}, \delta\underline{u} \in \mathbb{K}$$

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- Use optimization theory to convert to

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot\cdot \delta\nabla\underline{u}\, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C(\sigma_n, \sigma_t, g_n, \underset{\sim}{\xi}, \delta\underline{u})}}_{\text{Contact term}^*} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta\underline{u}\, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta\underline{u}\, d\Omega,$$

Unconstrained functional sub-spaces

$$\mathbb{L} = \left\{ \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \underline{u} = \underline{u}_0 \text{ on } \Gamma_u \right\}$$
$$\mathbb{K} = \left\{ \delta\underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta\underline{u} = 0 \text{ on } \Gamma_u \right\}$$

Contact term* is defined on the *potential contact zone* Γ_c^1.

[1] Duvaut, G. and Lions, J.L., 1972. *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972
[2] Duvant, G. and Lions, J.L., 1976. *Inequalities in mechanics and physics*, Springer

Optimization methods

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

Penalty method

- New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \left\langle -g(\mathbf{x}) \right\rangle^2} = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 \quad \textit{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 \quad \textit{contact} \end{cases}$$

  where ϵ is the penalty parameter.

- Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \left\langle -g(\mathbf{x}) \right\rangle \nabla g(\mathbf{x}) = 0$$

- Solution **tends** to the precise solution as $\epsilon \to \infty$

Lagrange multipliers method

Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \text{ReLU}(x)$

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2}$
- **Lagrange multipliers method**
  - New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$$

  - Saddle point problem

$$\min_x \max_\lambda \{\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})\} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x}) \geq 0} \{F(\mathbf{x})\}$$

  - Stationary point

$$\nabla_{\mathbf{x},\lambda} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_x g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0 \quad \text{need to verify} \quad \lambda \leq 0$$

- Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \mathrm{ReLU}(x)$

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2}$
- Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$
- **Augmented Lagrangian method**

  [Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]

  - New functional, augmented Lagrangian

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0, \ \textit{non-contact} \end{cases}$$

  - Stationary point

$$\nabla_{\mathbf{x},\lambda}\mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_x g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \textit{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \textit{if non-contact} \end{cases}$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \mathrm{ReLU}(x)$


Uzawa algorithm

Optimization methods

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2}$
- Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})}$
- **Augmented Lagrangian method: Uzawa algorithm**
  Iterative procedure for λ update:

$$\mathcal{L}_a(\mathbf{x}; \lambda_i) = F(\mathbf{x}) + \lambda_i g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), \text{ if } \lambda_i + 2\epsilon g(\mathbf{x}) \leq 0$$

$$\mathcal{L}_a(\mathbf{x}; \lambda_i) = F(\mathbf{x}) + (\lambda_i + \epsilon g(\mathbf{x}))\, g(\mathbf{x}), \text{ if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0$$

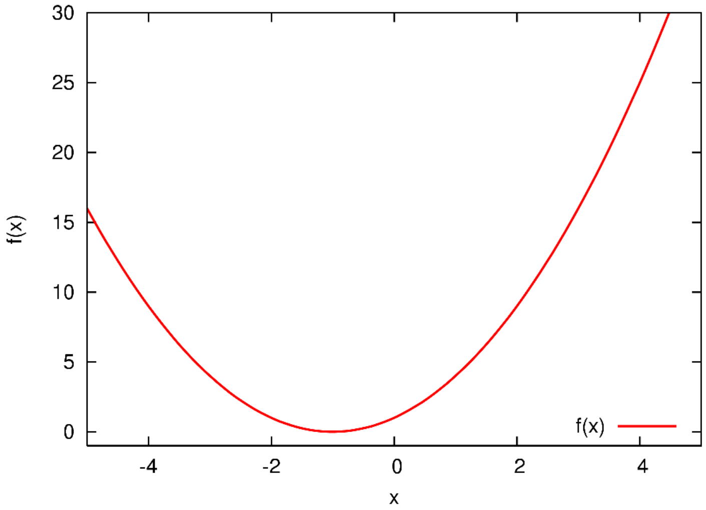
$$\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$$

$$\frac{\partial \mathcal{L}_a}{\partial x} = \frac{\partial F}{\partial x} + (\lambda_i + 2\epsilon g(x)) \frac{\partial g}{\partial x} = 0$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \mathrm{ReLU}(x)$


Uzawa algorithm

Optimization methods: example

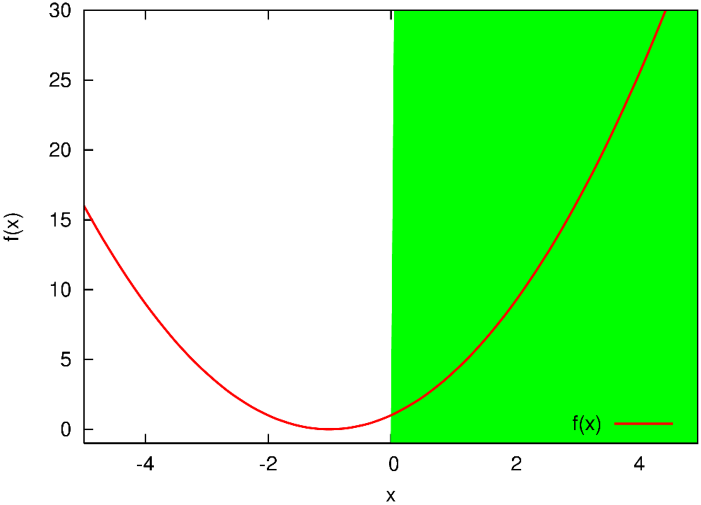


Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

Solution : $x^* = 0$

Optimization methods: example



Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

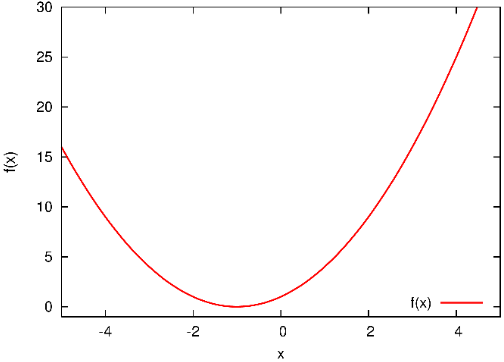
Solution : $x^* = 0$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$

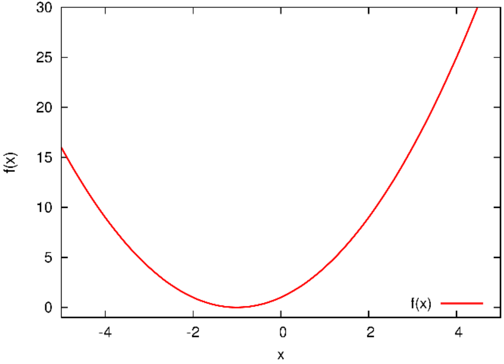


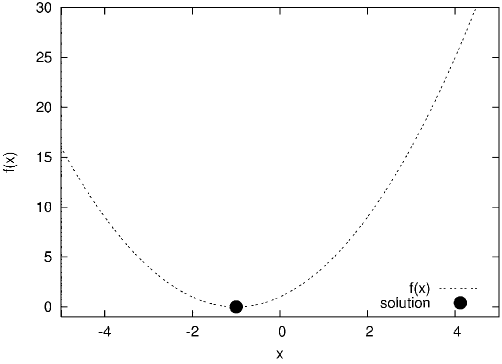
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





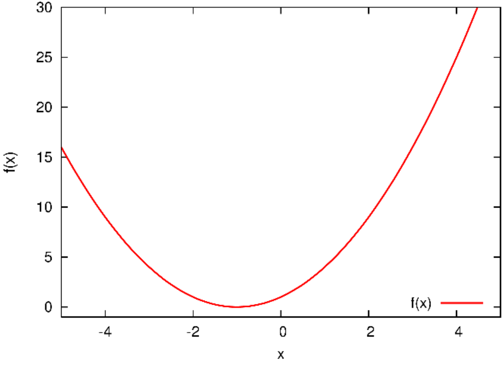
$\epsilon = 0$

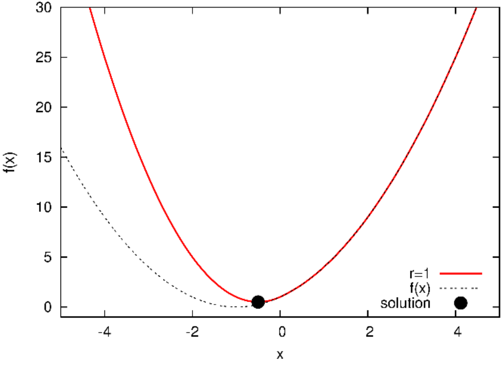
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \left\langle -g(x) \right\rangle^2}$$





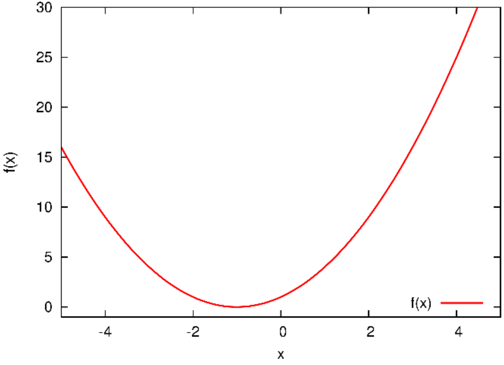
$\epsilon = 1$

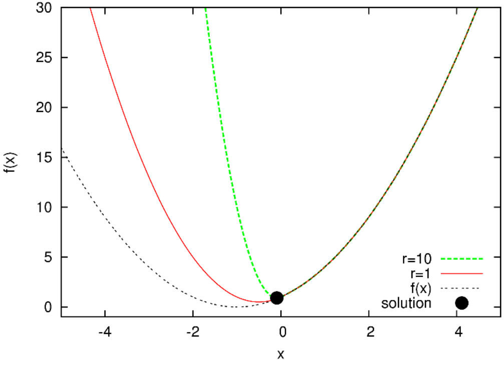
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$





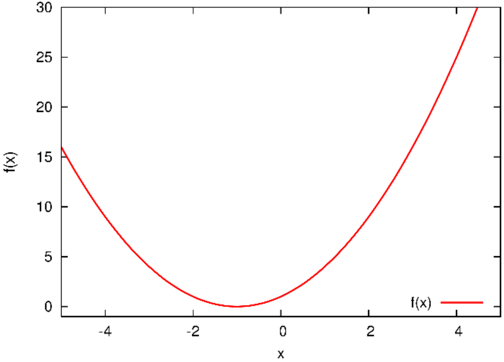
$\epsilon = 10$

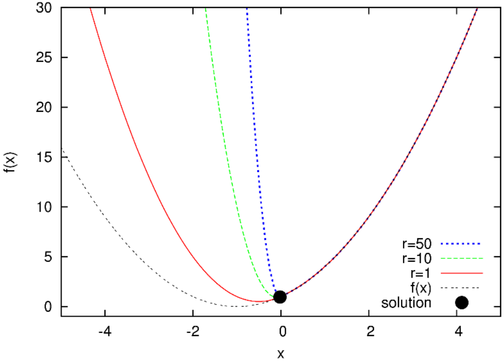
Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \left\langle -g(x) \right\rangle^2}$$





$\epsilon = 50$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Penalty method**

$$F_p(x) = F(x) + \boxed{\epsilon \langle -g(x) \rangle^2}$$

Advantages ☺

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- "mathematically" smooth functional

Drawbacks ☹

- practically non-smooth functional
- solution is not exact:
  - too small penalty → large penetration
  - too large penalty → ill-conditioning of the tangent matrix
- user has to choose penalty ϵ properly or automatically and/or adapt during convergence

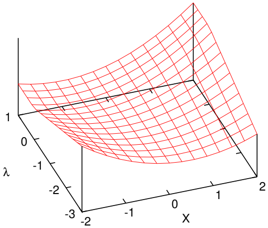
Lagrange multipliers method: example

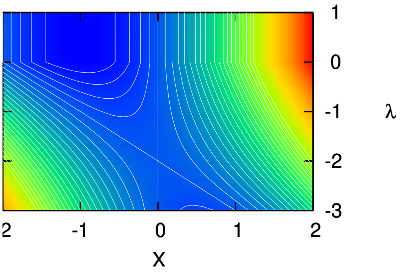
$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Lagrange multipliers method**

$\mathcal{L}(x, \lambda) = F(x) + \boxed{\lambda g(x)} \longrightarrow$ Saddle point $\longrightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda)$

Need to check that $\lambda \leq 0$





Lagrange multipliers method: example

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  Need to check that $\lambda \leq 0$

Advantages ☺

- exact solution
- no adjustable parameters

Drawbacks ☹

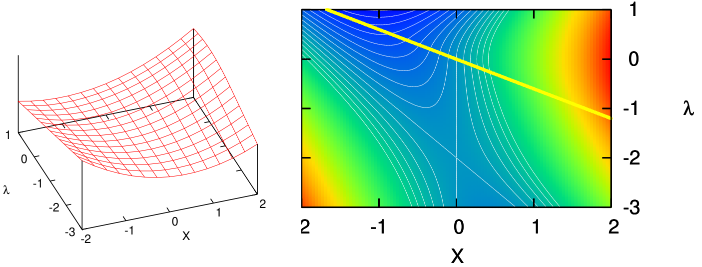
- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained: $\lambda \leq 0$

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \ \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0, \ \textit{non-contact} \end{cases}$$



Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

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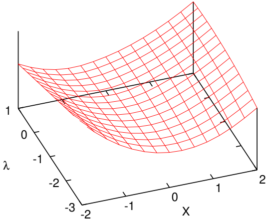

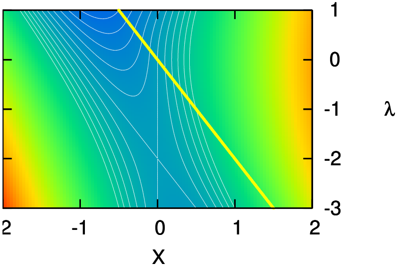

Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

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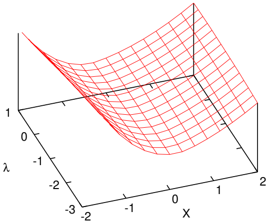
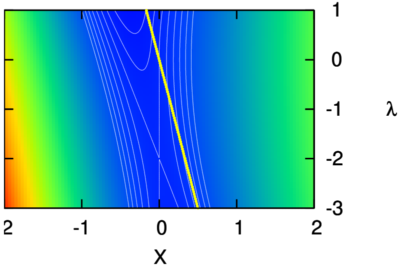




Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \; \textit{contact} \\ -\frac{1}{4\epsilon}\lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0, \; \textit{non-contact} \end{cases}$$

Advantages ☺

- exact solution
- smoother functional (!)
- fully unconstrained

Drawbacks ☹

- additional degrees of freedom
- quite sensitive to parameter ϵ
- need to adjust ϵ during convergence

Augmented Lagrangian with Uzawa algorithm

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x};\lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, \quad \text{if } \lambda + 2\epsilon g(\mathbf{x}) \le 0, \ \textit{contact}$$

Fix $\lambda = \lambda_0$

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + \lambda_0 g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), \text{ if } \lambda_0 + 2\epsilon g(\mathbf{x}) \le 0$$

Converge with respect to x

Augmented Lagrangian with Uzawa algorithm

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x};\lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, \quad \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0,\ \textit{contact}$$

Fix $\lambda = \lambda_0$

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + [\lambda_0 + \epsilon g(\mathbf{x})]\ g(\mathbf{x}),\ \text{if } \lambda_0 + 2\epsilon g(\mathbf{x}) \leq 0$$

Converge with respect to x and update $\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$

Augmented Lagrangian with Uzawa algorithm

- **Augmented Lagrangian method**

$$\mathcal{L}_a(\mathbf{x};\lambda) = F(\mathbf{x}) + \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, \quad \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0,\ \textit{contact}$$

Fix $\lambda = \lambda_0$
Converge with respect to x and update $\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$

$$\mathcal{L}_a(\mathbf{x},\lambda) = F(\mathbf{x}) + [\lambda_1 + \epsilon g(\mathbf{x})]\ g(\mathbf{x}),\ \text{if } \lambda_1 + 2\epsilon g(\mathbf{x}) \leq 0$$

Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

$$\underline{u} \in \mathbb{L}, \delta \underline{u} \in \mathbb{K}, \qquad \mathbb{L} = \left\{ \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \underline{u} = \underline{u}_0 \text{ on } \Gamma_u \right\}, \quad \mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \text{ on } \Gamma_u \right\}$$

- Penalty method

Pressure: $\sigma_n = \epsilon g_n$, Shear: $\underline{\sigma}_t = \begin{cases} \epsilon \underline{g}_t, & \textit{if stick } |\sigma_t| < \mu |\sigma_n| \\ \mu \epsilon g_n \delta \underline{g}_t / |\delta \underline{g}_t|, & \textit{if slip } |\sigma_t| = \mu |\sigma_n| \end{cases}$

Contact term

$$C = C(g_n, \underline{g}_t, \delta g_n, \delta \underline{g}_t) = \sigma_n \delta g_n + \underline{\sigma}_t \cdot \delta \underline{g}_t$$

Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} \, d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} \, d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} \, d\Omega,$$

$$\underline{u} \in \mathbb{L}, \delta \underline{u} \in \mathbb{K}, \qquad \mathbb{L} = \left\{ \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \underline{u} = \underline{u}_0 \;\text{ on } \Gamma_u \right\}, \quad \mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \;\middle|\; \delta \underline{u} = 0 \;\text{ on } \Gamma_u \right\}$$

- Augmented Lagrangian method

Contact term

$$C = C(g_n, \underline{g}_t, \lambda_n, \underline{\lambda}_t, \delta g_n, \delta \underline{g}_t, \delta \lambda_n, \delta \underline{\lambda}_t)$$

$$C = \begin{cases} -\frac{1}{\epsilon} \left(\lambda_n \delta \lambda_n - \underline{\lambda}_t \cdot \delta \underline{\lambda}_t \right), & \textit{if non-contact } \lambda_n + \epsilon g_n \geq 0 \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \hat{\underline{\lambda}}_t \cdot \delta \underline{g}_t + \underline{g}_t \cdot \delta \hat{\underline{\lambda}}_t, & \textit{if stick } |\hat{\underline{\lambda}}_t| \leq \mu |\hat{\sigma}_n| \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \mu \hat{\sigma}_n - \mu \hat{\sigma}_n \dfrac{\hat{\underline{\lambda}}_t}{|\hat{\underline{\lambda}}_t|} \cdot \delta \underline{g}_t - \frac{1}{\epsilon} \left(\lambda_t + \mu \hat{\sigma}_n \dfrac{\hat{\underline{\lambda}}_t}{|\hat{\underline{\lambda}}_t|} \right) \cdot \delta \underline{\lambda}_t, & \textit{if slip } |\hat{\underline{\lambda}}_t| \geq \mu |\hat{\sigma}_n| \end{cases}$$

where $\hat{\lambda}_n = \lambda_n + \epsilon g_n$ and $\hat{\underline{\lambda}}_t = \underline{\lambda}_t + \epsilon \underline{g}_t$.

Application to contact problems: linearization

- Non-linear equation

$$R(\underline{u},\underline{f}) = 0$$

- Contains $\delta g_n, \delta \underline{g}_t$
- Use Newton-Raphson method
- Initial state at step i

$$R(\underline{u}^i,\underline{f}^i) = 0$$

- Should be also satisfied at step $i+1$

$$R(\underline{u}^{i+1},\underline{f}^{i+1}) = R(\underline{u}^i + \delta\underline{u},\underline{f}^{i+1}) = 0$$

- Linearize

$$R(\underline{u}^i + \delta\underline{u},\underline{f}^{i+1}) = R(\underline{u}^i,\underline{f}^{i+1}) + \frac{\partial R(\underline{u})}{\partial \underline{u}}\delta\underline{u} = 0$$

- Finally

$$\delta\underline{u} = -\underbrace{\left[\frac{\partial R(\underline{u})}{\partial \underline{u}}\right]^{-1}}_{\text{contains } \Delta\delta g_n,\, \Delta\delta \underline{g}_t} R(\underline{u}^i)$$

- NB: Contact problem does not satisfy conditions of Kantorovich theorem on the convergence of Newton's method.

Variation of geometrical quantities

Normal gap

- First variation enters in the residual vector:

$$\delta g_n = \underline{\boldsymbol{n}} \cdot (\delta \underline{\boldsymbol{r}}_s - \delta \underline{\boldsymbol{\rho}})$$

- Second variation enters in the tangent matrix:

$$\Delta\delta g_n = -\underline{\boldsymbol{n}} \cdot \left(\delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \Delta \underset{\sim}{\xi} + \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \delta \underset{\sim}{\xi} \right) - \Delta \underset{\sim}{\xi}^T \underset{\approx}{\mathrm{H}} \, \delta \underset{\sim}{\xi} + \\ + g_n \left(\Delta \underset{\sim}{\xi}^T \underset{\approx}{\mathrm{H}} + \underline{\boldsymbol{n}} \cdot \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \right) \underset{\approx}{\bar{\mathrm{A}}} \left(\underline{\boldsymbol{n}} \cdot \delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \underset{\approx}{\mathrm{H}} \, \delta \underset{\sim}{\xi} \right)$$

Variation of geometrical quantities

Convective coordinate of the projection

- First variation enters in the residual vector:

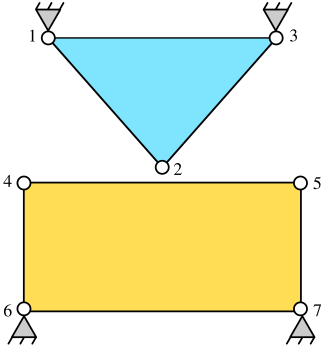
$$\delta \underset{\sim}{\xi} = \left[\underset{\approx}{A} - g_n \underset{\approx}{H} \right]^{-1} \left(\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \cdot (\delta \underline{\boldsymbol{r}}_s - \delta \underline{\rho}) + g_n \underline{\boldsymbol{n}} \cdot \delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \right)$$

- Second variation enters in the tangent matrix:

$$\begin{aligned}
\Delta\delta \underset{\sim}{\xi} = (g_n \underset{\approx}{H} - \underset{\approx}{A})^{-1} \Bigg\{ & \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \cdot \left(\delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \Delta \underset{\sim}{\xi} + \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \delta \underset{\sim}{\xi} \right) + \Delta \underset{\sim}{\xi}^T \left(\frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} \cdot \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \right) \delta \underset{\sim}{\xi} - \\
& - g_n \underline{\boldsymbol{n}} \cdot \left(\delta \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \Delta \underset{\sim}{\xi} + \Delta \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \delta \underset{\sim}{\xi} \right) - g_n \Delta \underset{\sim}{\xi}^T \left(\underline{\boldsymbol{n}} \cdot \frac{\partial^3 \underline{\rho}}{\partial \underset{\sim}{\xi}^3} \right) \delta \underset{\sim}{\xi} + \\
& + \left[g_n \left(\delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \delta \underset{\sim}{\xi} \right) \cdot \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \bar{\underset{\approx}{A}} - \delta g_n \underset{\approx}{I} \right] \left(\underline{\boldsymbol{n}} \cdot \Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \underset{\approx}{H} \Delta \underset{\sim}{\xi} \right) + \\
& + \left[g_n \left(\Delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \frac{\partial^2 \underline{\rho}}{\partial \underset{\sim}{\xi}^2} \Delta \underset{\sim}{\xi} \right) \cdot \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}}^T \bar{\underset{\approx}{A}} - \Delta g_n \underset{\approx}{I} \right] \left(\underline{\boldsymbol{n}} \cdot \delta \frac{\partial \underline{\rho}}{\partial \underset{\sim}{\xi}} + \underset{\approx}{H} \delta \underset{\sim}{\xi} \right) \Bigg\}
\end{aligned}$$

Example

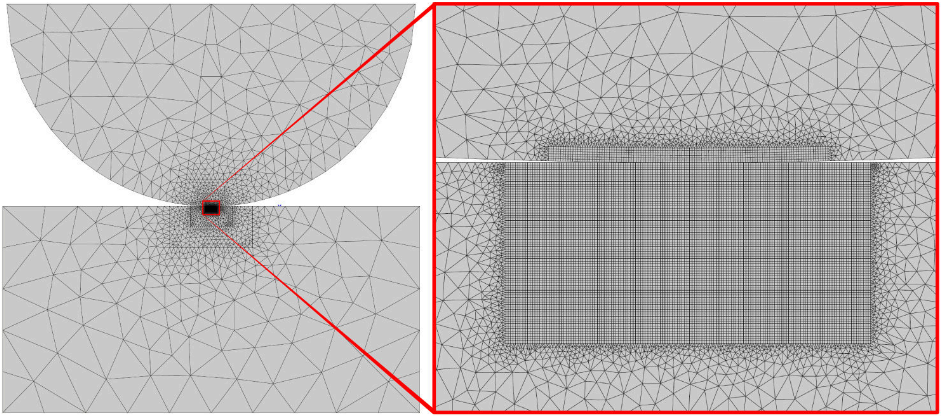
- Use penalty method to enforce Dirichlet BC
- Use penalty method to enforce contact constraints
- First, detect contact elements
- Second, construct updated residual vector and tangent matrix



Contact between two elements

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones



Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]

2D ~ 30 000 DoFs,   3D ~ 5 000 000 DoFs

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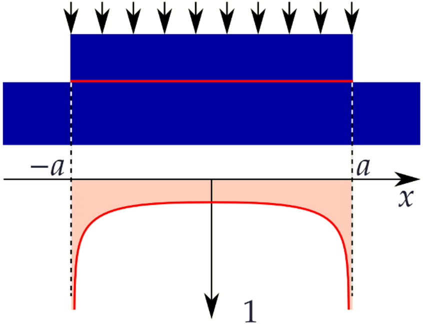
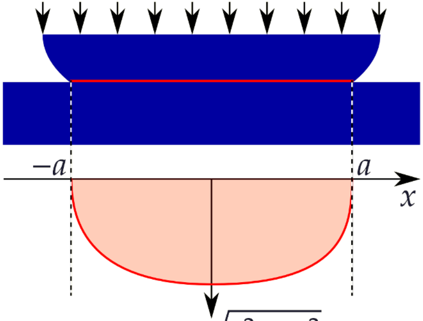


$$\sigma_n \sim \frac{1}{\sqrt{a^2 - x^2}}$$

$$\sigma_n \xrightarrow[x \to a]{} -\infty \qquad \left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x \to a]{} \infty$$



$$\sigma_n \sim \sqrt{a^2 - x^2}$$

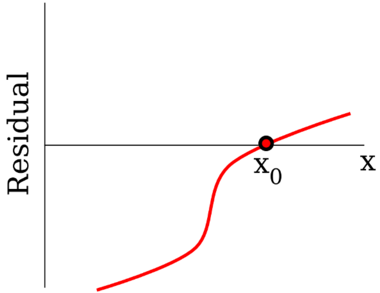
$$\left|\frac{\partial \sigma_n}{\partial x}\right| \xrightarrow[x \to a]{} \infty$$

Infinite contact pressure and/or its derivative

Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
  - strong non-linearities of contact / friction problems
  - non-uniqueness of solution for frictional problems

Infinite looping



Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

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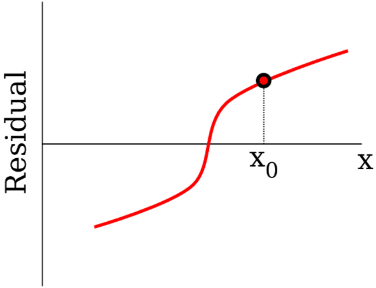

Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

Particularities: mesh and convergence

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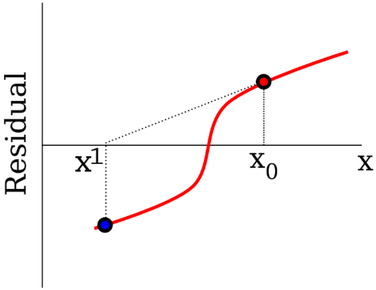

Iterations of Newton-Raphson method $R(x_0, f_1) + \left.\frac{\partial R}{\partial x}\right|_{x_0} \delta x = 0 \rightarrow \delta x = -\left.\frac{\partial R}{\partial x}\right|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$

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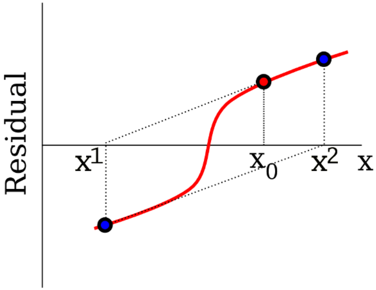

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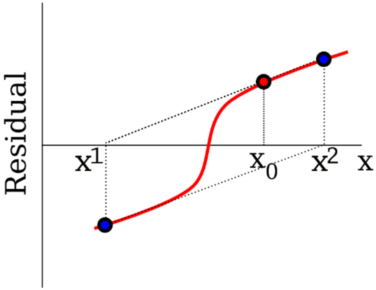

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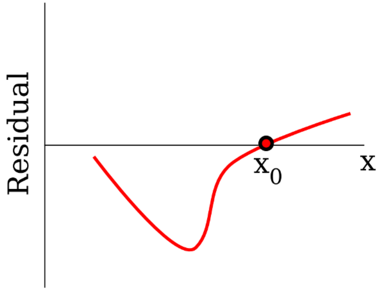
Convergence to a "false" solution



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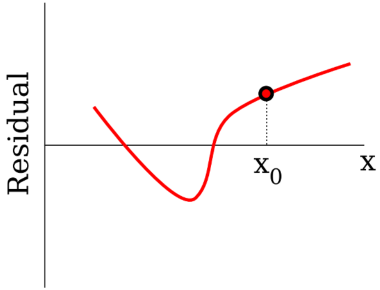

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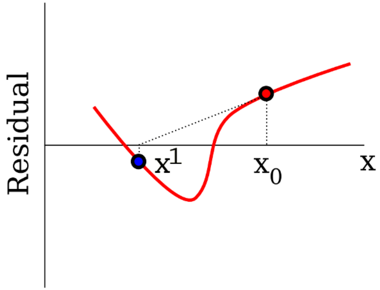

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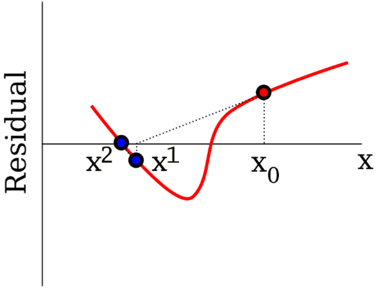

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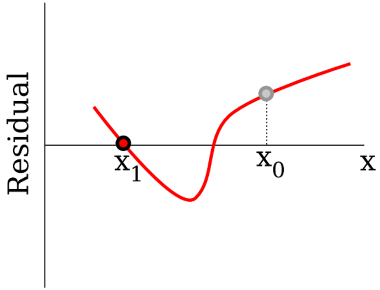
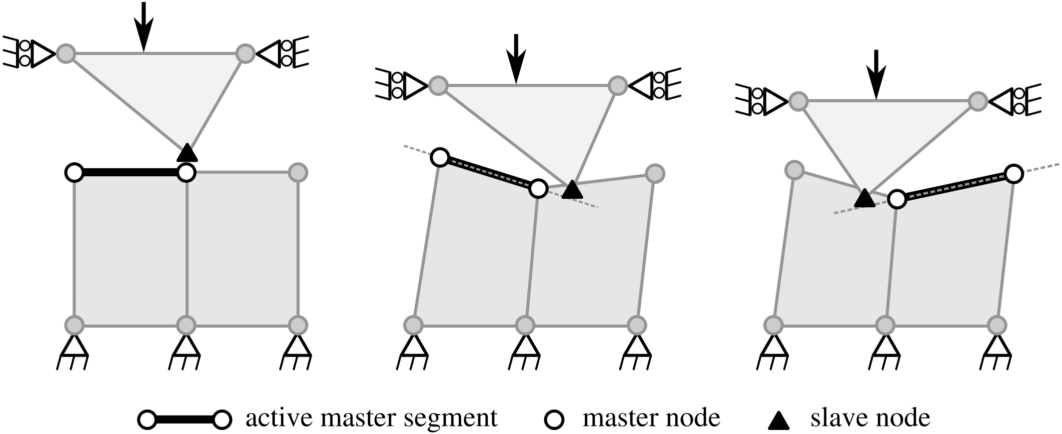


Convergence, but is it a "true" solution ?

Convergence problems: examples

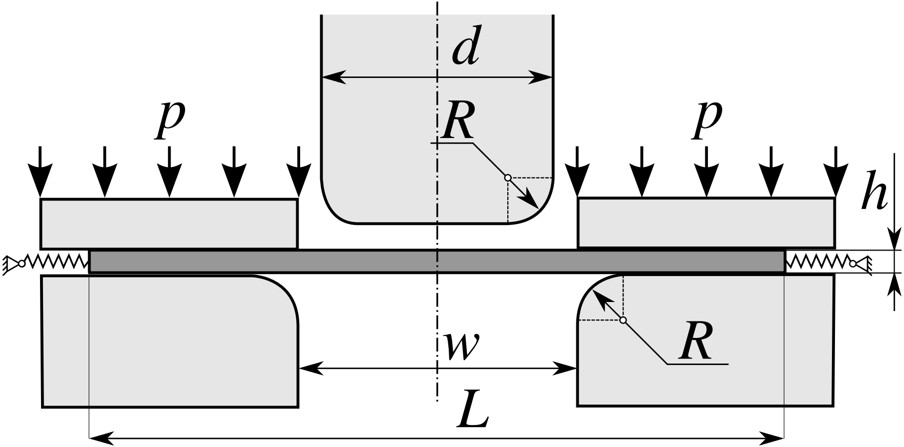
- Infinite looping, e.g.



- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients[1]
- Combination of non-linearities (e.g., plasticity+contact)

Convergence problems: examples

- Simulation of a deep drawing problem
- Finite strain plasticity + frictional contact



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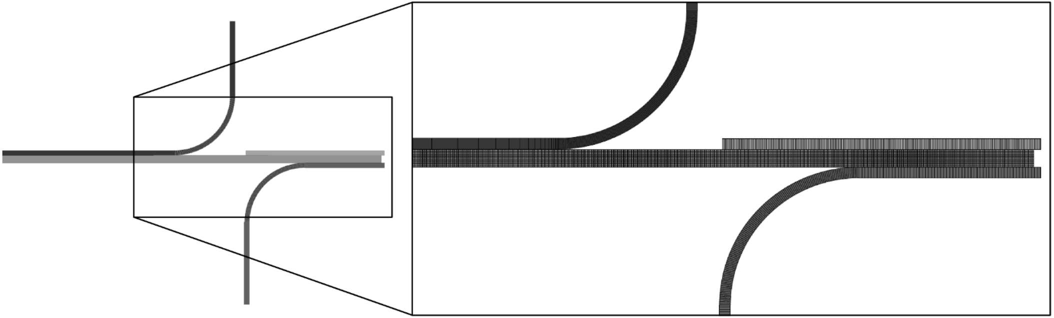

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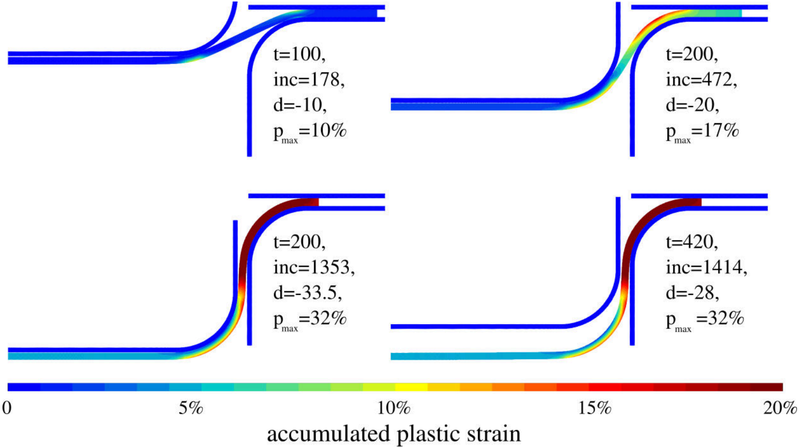

Convergence problems: examples

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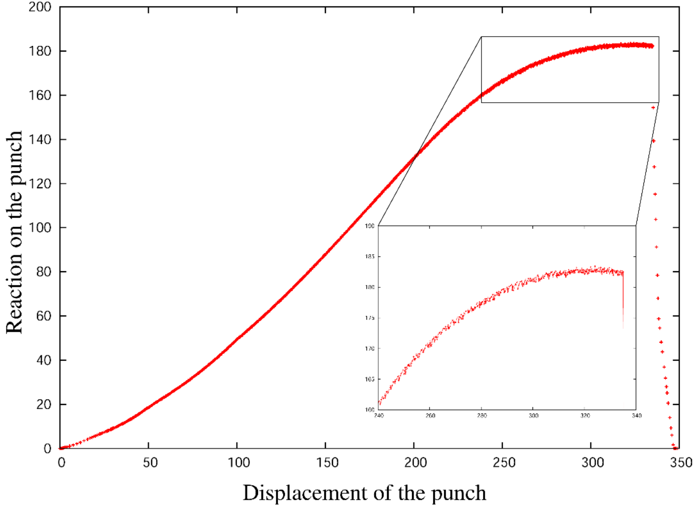
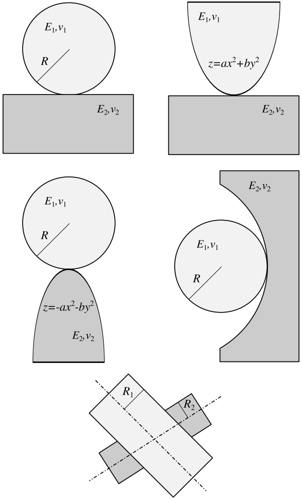


Cylinder-plane frictional contact

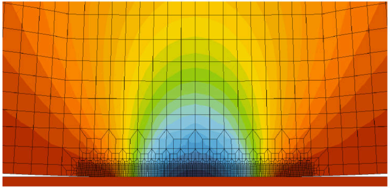
- Non-conservative problem, history of loading is crucial



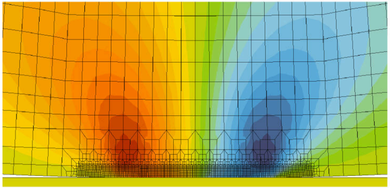
Cylinder-plane frictional contact

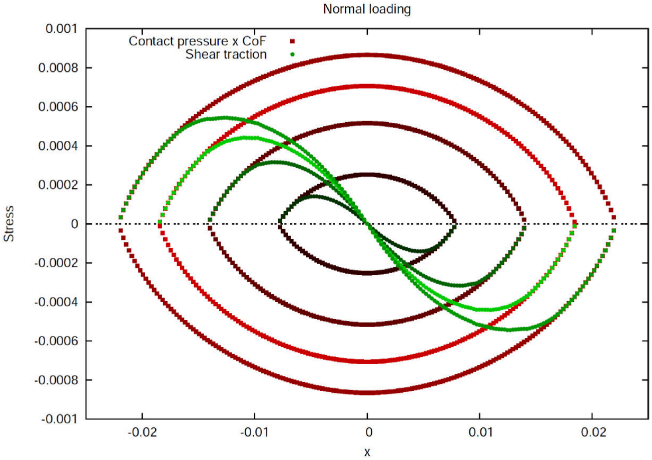
- Non-conservative problem, history of loading is crucial

sig22 at maximal normal load



sig12 at maximal normal load





Press in 100 increments, $u_z \sim t^2$

Cylinder-plane frictional contact

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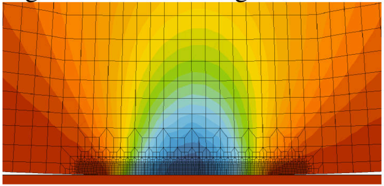
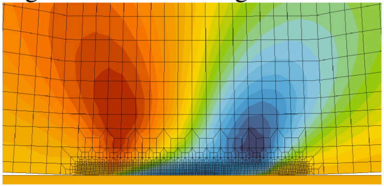
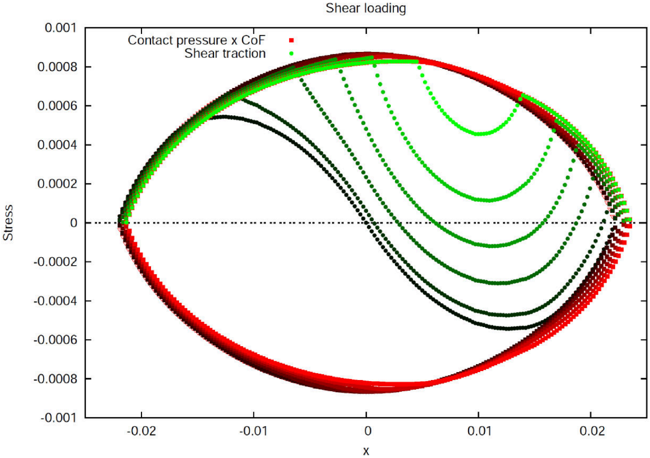
sig22 at maximal tangential load



sig12 at maximal tangential load

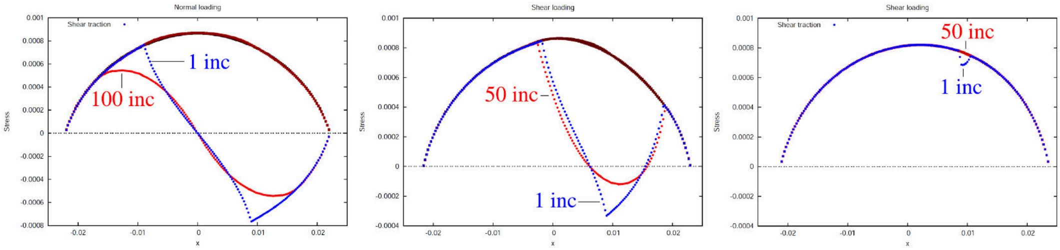




Shift in 100 increments, $u_z \sim t$

Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

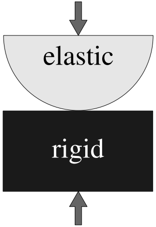


Comparison with: press in 1 increment, shift in 2 increments

Before sticking, every point of the contact interface has to pass through the slip zone. It is impossible when loaded too fast.

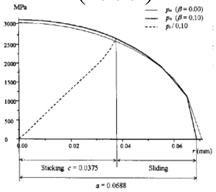
Warning friction!

- For dissimilar materials, the *friction matters* even in normal contact
- The problem is thus path-dependent, the B.C. should be changed slowly

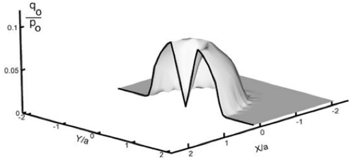


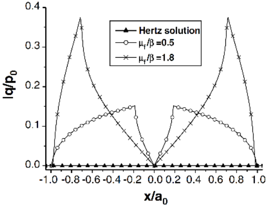
Erroneous solutions

(1999)

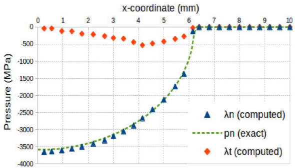


(2003)



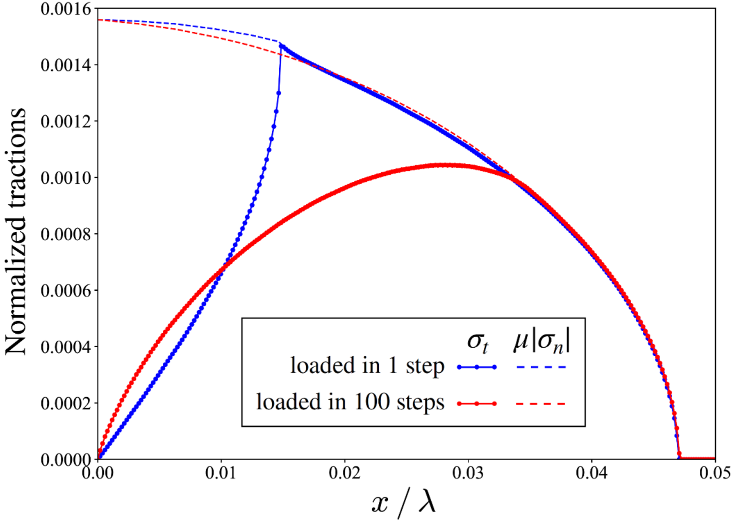


(2008)



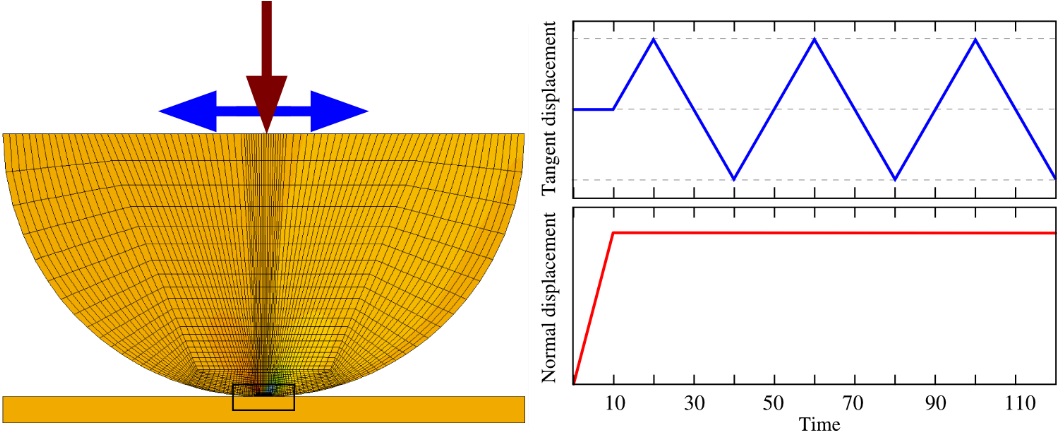
(2017)

Warning friction!



[1] A.G. Shvarts, PhD thesis, MINES ParisTech (2019)

Sphere-plane frictional contact: cycling



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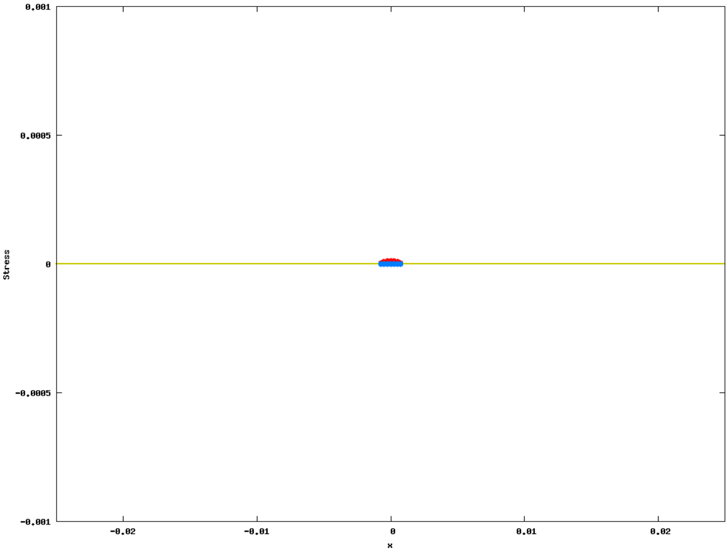

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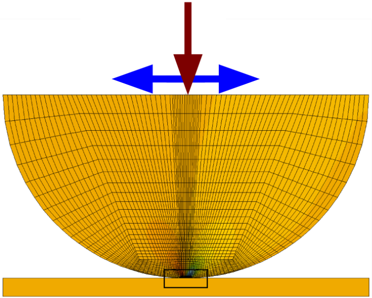
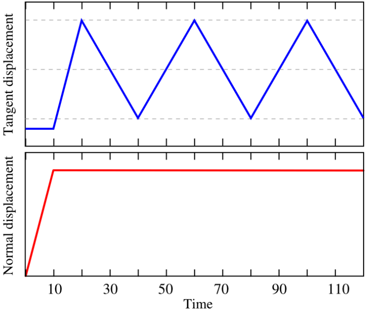
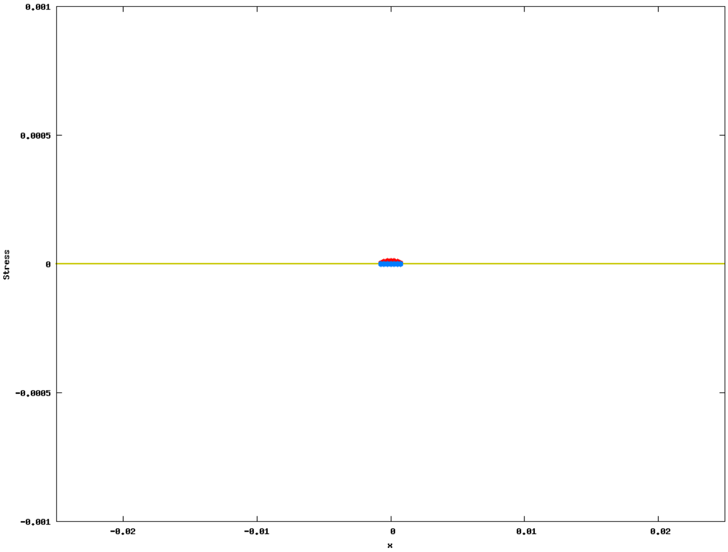


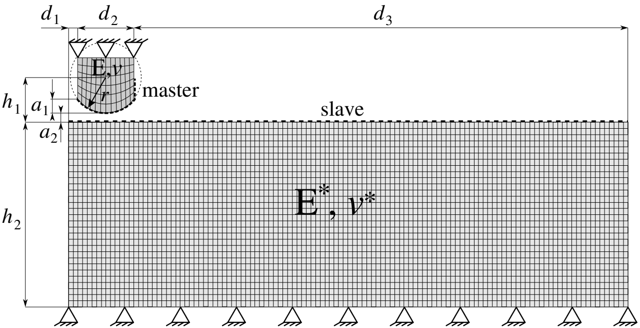


Sphere-plane frictional contact: cycling



Shallow ironing test

- Deformable-on-deformable frictional sliding
- Results obtained by different groups[1,2,3,4,5,6] differ significantly
- Local and global friction coefficients may differ



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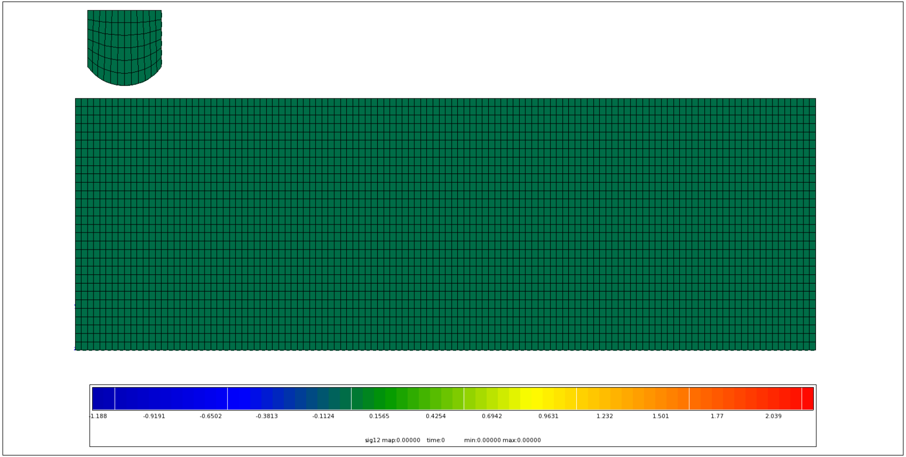
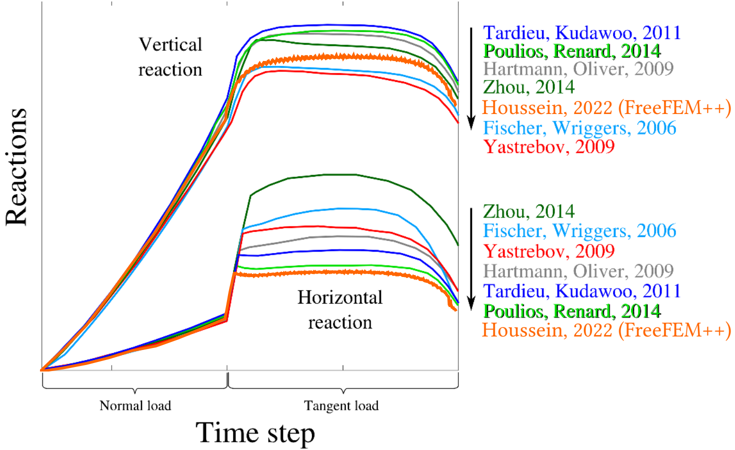
[6] Zhou Lei's blog, http://kt2008plus.blogspot.de

Shallow ironing test



Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



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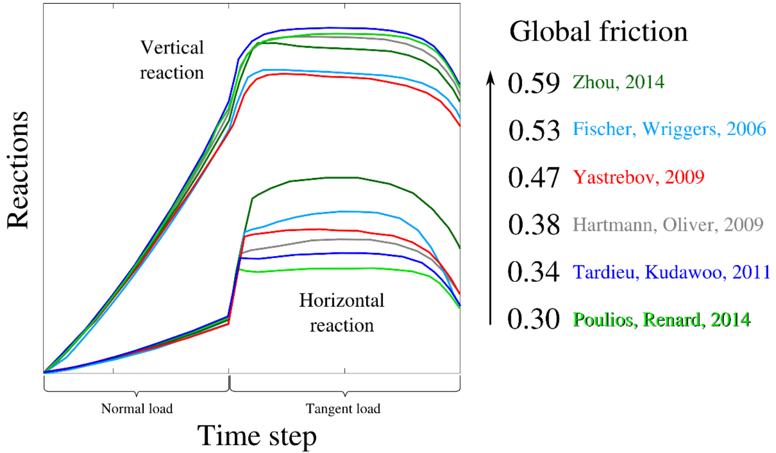
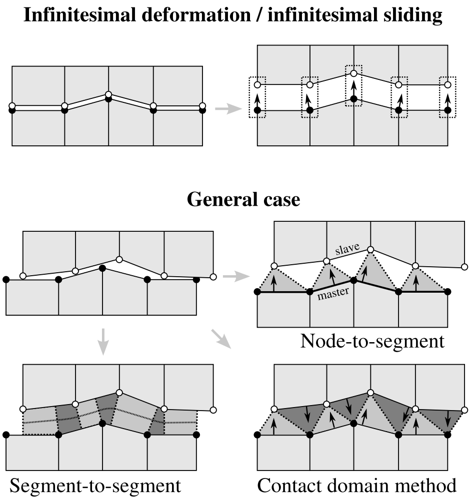


Local coefficient of friction $\mu_l = 0.3$

Reading

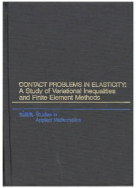
- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics



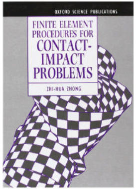
Contact discretization techniques

Reading

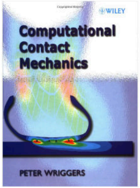
- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics



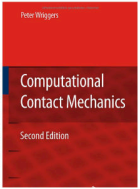
Kikuchi, Oden (1988)



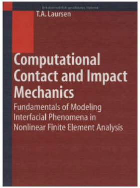
Zhong (1993)



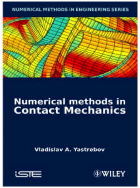
Wriggers (2002)



Wriggers, 2nd ed. (2006)



Laursen (2002)



Yastrebov (2013)

Merci de votre attention!