

Contact mechanics and elements of tribology

Lecture 6.

Computational Contact Mechanics

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Evry, France*

@ Centre des Matériaux
February 13, 2019

- Introduction
- Governing equations
- Optimization methods
- Resolution algorithm
- Examples

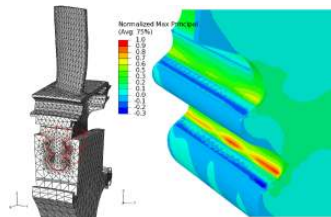
Industrial and natural contact problems

1 Assembled parts, e.g. engines



Aircraft's engine GP 7200

www.safran-group.com



[1] M. W. R. Savage

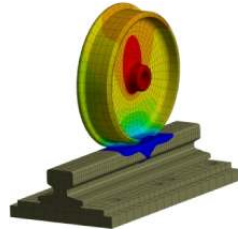
J. Eng. Gas Turb. Power, 134:012501 (2012)

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts



High speed train TGV www.sncf.com



Wilde/ANSYS wildeanalysis.co.uk

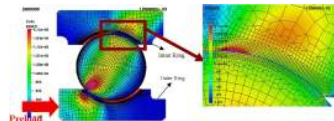
Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings



Bearings

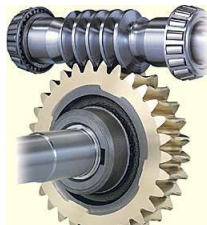
www.skf.com



[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier
Mech. Syst. Signal Pr., 24:1068-1080 (2010)

Industrial and natural contact problems

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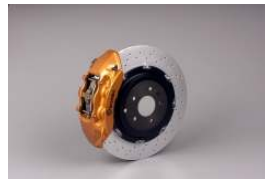
Helical gear www.tpg.com.tw



www.mscsoftware.com

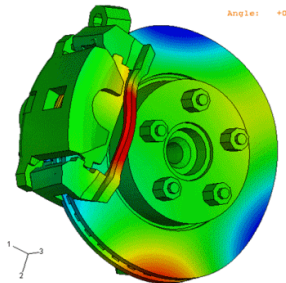
Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
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- 4 Breaking systems



Assembled breaking system

www.brembo.com



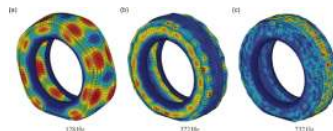
www.mechanicalengineeringblog.com

Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems
- 5 Tire-road contact



Tire-road contact www.michelin.com



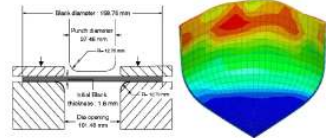
[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, J. Sound Vib., 309:20-39 (2008)

Industrial and natural contact problems

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- 4 Breaking systems
- 5 Tire-road contact
- 6 Metal forming



Deep drawing www.thomasnet.com



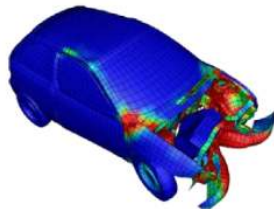
[1] G. Rousselier, F. Barlat, J. W. Yoon
Int. J. Plasticity, 25:2383-2409 (2009)

Industrial and natural contact problems

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- 7 Crash tests



Crash-test www.porsche.com



[1] O. Klyavin, A. Michailov, A. Borovkov
www.fea.ru

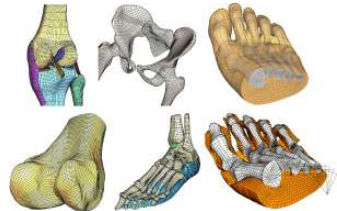
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- 8 Biomechanics



Human articulations

www.sportssupplements.net



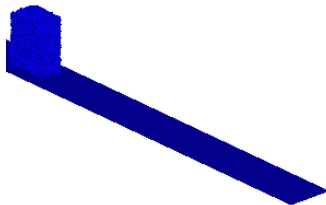
*J. A. Weiss, University of Utah
Musculoskeletal Research Laboratories*

Industrial and natural contact problems

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- 9 Granular materials



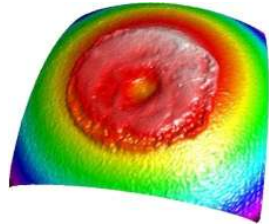
Sand dunes www.en.wikipedia.org



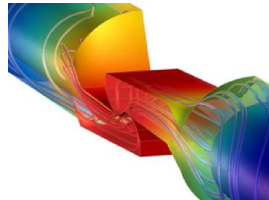
E. Azema et al, LMGC90

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- 8 Biomechanics
- 9 Granular materials
- 10 Electric contacts



Damage at electric contact zone
www.taicaan.com



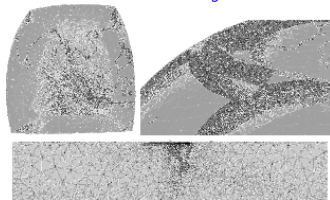
Simulation of electric current
www.comsol.com

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- 11 Tectonic motions



San-Andreas fault, by M. Rightmire
www.sciencedude.ocregister.com



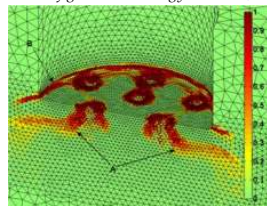
[1] J.D. Garaud, L. Fleitout, G. Caillaud
Colloque CSMA (2009)

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- 11 Tectonic motions
- 12 Deep drilling



Drill Bit tool [RobitRocktools](#);
extraction of geothermal energy ([SINTEF, NTNU](#))



[1] T. Saksala, *Int. J. Numer. Anal. Meth. Geomech.* (2012)

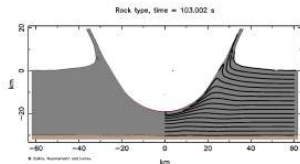
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- 13 Impact and fragmentation



Impact crater, Arizona

www.MrEclipse.com et maps.google.com



*Simulation of formation of Copernicus crater
Yue Z., Johnson B. C., et al. Projectile*

remnants in central peaks of lunar impact
craters. *Nature Geo* 6 (2013)

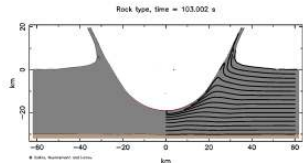
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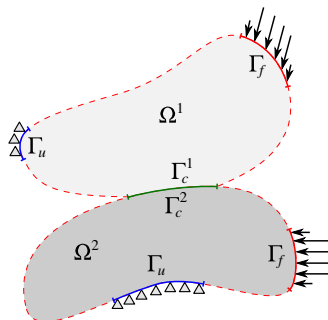
Equilibrium and contact conditions

■ Balance of momentum

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{\underline{n}} = \underline{\underline{t}}_0 & \text{on } \Gamma_f \\ \underline{\underline{u}} = \underline{\underline{u}}_0 & \text{on } \Gamma_u \\ ? & \text{on } \Gamma_c \end{cases}$$

■ Frictionless contact conditions (*intuitive*)

- 1 No penetration
- 2 No adhesion
- 3 No shear transfer



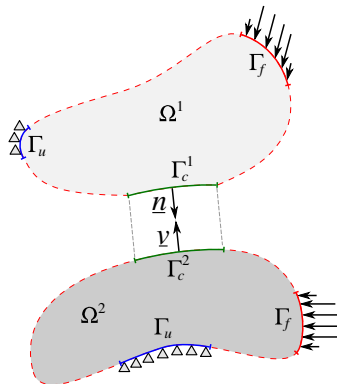
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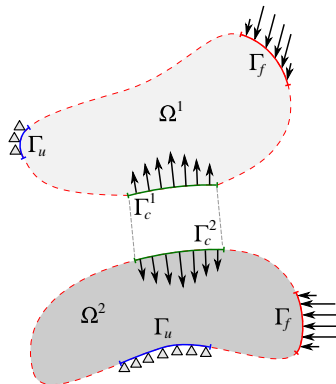
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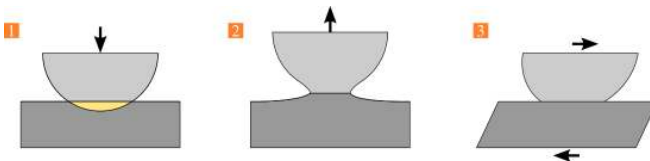
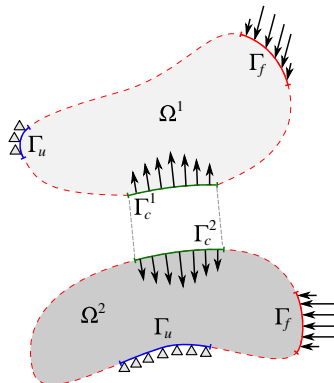
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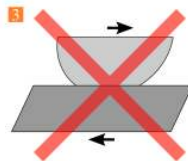
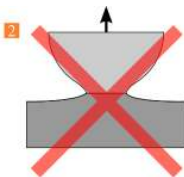
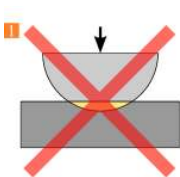
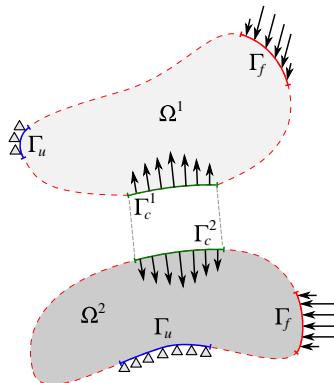
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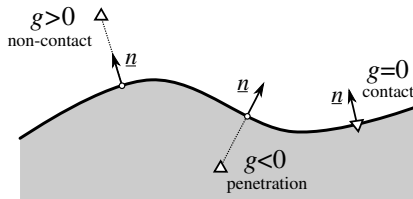


■ Gap function g

- gap = - penetration
- asymmetric function
- defined for
 - separation $g > 0$
 - contact $g = 0$
 - penetration $g < 0$
- governs normal contact

■ Master and slave split

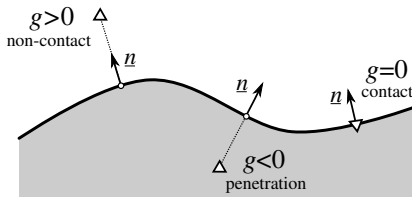
Gap function is determined for all slave points with respect to the master surface



Gap between a slave point and a master surface

■ Gap function g

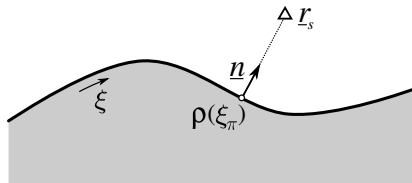
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Gap between a slave point and a master surface

■ Master and slave split

Gap function is determined for all slave points with respect to the master surface



Definition of the normal gap

Consider existence and uniqueness



$g_n = \underline{n} \cdot [\underline{r}_s - \underline{\rho}(\xi_\pi)]$,
 $\underline{n}$ is a unit normal vector, $\underline{r}_s$
 slave point, $\underline{\rho}(\xi_\pi)$ projection
 point at master surface

Frictionless or normal contact conditions

■ No penetration

Always non-negative gap

$$g \geq 0$$

■ No adhesion

Always non-positive contact pressure

$$\sigma_n^* \leq 0$$

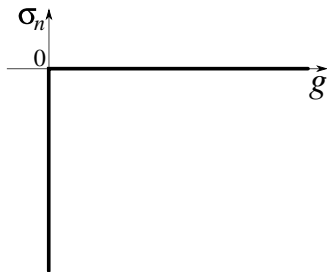
■ Complementary condition

Either zero gap and non-zero pressure, or non-zero gap and zero pressure

$$g \sigma_n = 0$$

■ No shear transfer (automatically)

$$\underline{\sigma}_t^* = 0$$



Scheme explaining normal contact conditions

$$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$$

$$\underline{\sigma}_t^* = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{I} - \underline{n} \otimes \underline{n})$$

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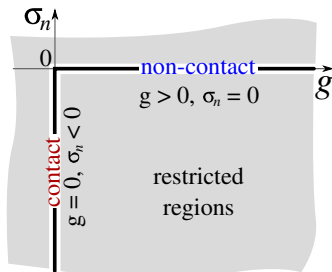
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Improved scheme explaining normal contact conditions

$$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$$

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Frictionless or normal contact conditions

In mechanics:

Normal contact conditions

$\equiv$

Frictionless contact conditions

$\equiv$

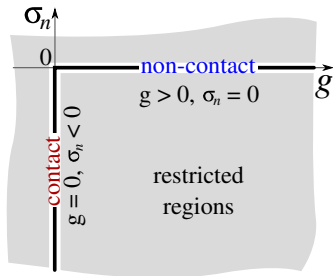
Hertz¹-Signorini^[2] conditions

$\equiv$

Hertz¹-Signorini^[2]-Moreau^[3] conditions

also known in **optimization theory** as

Karush^[4]-Kuhn^[5]-Tucker^[6] conditions



Improved scheme explaining normal contact conditions

$$g \geq 0, \quad \sigma_n \leq 0, \quad g\sigma_n = 0$$

¹Heinrich Rudolf Hertz (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

²Antonio Signorini (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

³Jean Jacques Moreau (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

⁴William Karush (1917–1997), ⁵Harold William Kuhn (1925) American mathematicians,

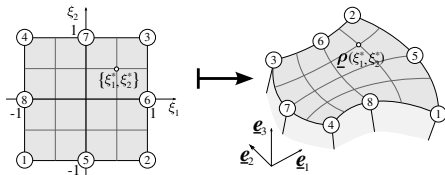
⁶Albert William Tucker (1905–1995) a Canadian mathematician.

Relative sliding

Recall:

- Convective coordinate in parent space $\xi_i \in (-1; 1)$
- Mapping to real space

$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^8 N^i(\xi_1, \xi_2) \underline{\rho}^i(t)$$

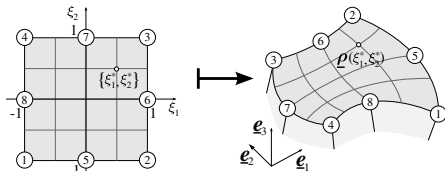


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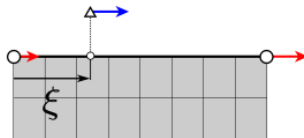
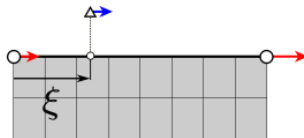
■ Tangential slip velocity $\underline{v}_t$

must take into account:

- only tangential component
- relative rigid body motion
- master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1} \dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2} \dot{\xi}_2$$

where $\partial \underline{\rho} / \partial \xi_i$ are the tangent vectors of the local basis and ξ_i are the convective coordinates.



Relative slip between a slave point and a deformable master surface

Relative sliding: example

Consider a one-dimensional example:

P is a projection of A on segment BC .

$$x_P = \xi x_C + (1 - \xi)x_B \quad (1)$$

Velocity of the projection point

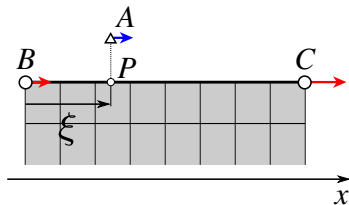
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1 - \xi) \dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B) \dot{\xi}}_{\frac{\partial x_P}{\partial \xi} \dot{\xi}}$$

Subtract the velocity of point x_P for fixed ξ

$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B) \dot{\xi} = \frac{\partial x}{\partial \xi} \dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left. \frac{\partial x}{\partial \xi} \right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

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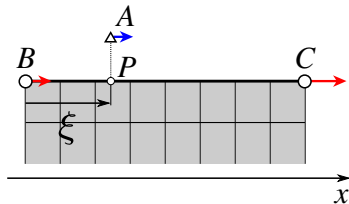
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Example of a one-dimensional relative slip



Ship-river analogy

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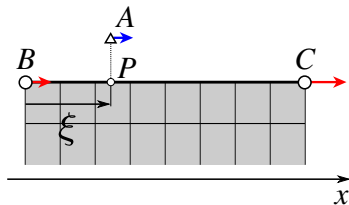
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$$x_P = \xi x_C + (1 - \xi)x_B \quad (1)$$

Velocity of the projection point

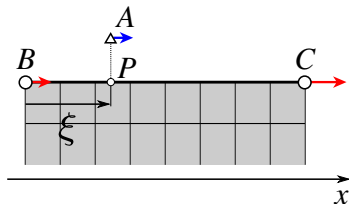
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1 - \xi) \dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B) \dot{\xi}}_{\frac{\partial x_P}{\partial \xi} \dot{\xi}}$$

Subtract the velocity of point x_P for fixed ξ

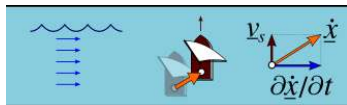
$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B) \dot{\xi} = \frac{\partial x}{\partial \xi} \dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left. \frac{\partial x}{\partial \xi} \right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip



Ship-river analogy

Li derivative: the change of a vector field along the change of another vector field

Amontons-Coulomb's friction

- **No contact** $g > 0, \sigma_n = 0$

- **Stick** $|\underline{v}_t| = 0$

Inside slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

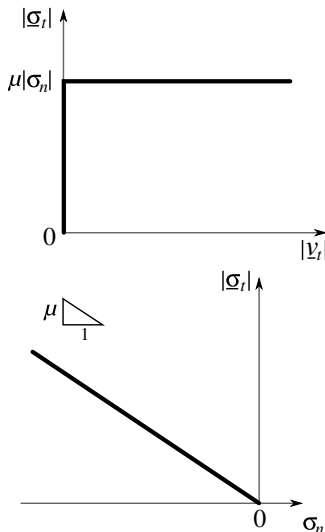
On slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

- **Complementary condition**

Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

$$|\underline{v}_t| \left(|\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$



Scheme explaining frictional contact conditions

Amontons-Coulomb's friction

- **No contact** $g > 0, \sigma_n = 0$

- **Stick** $|\underline{v}_t| = 0$

Inside slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

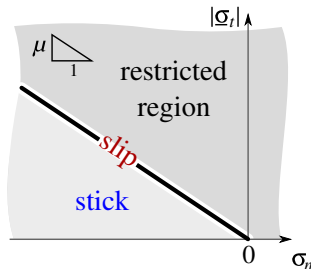
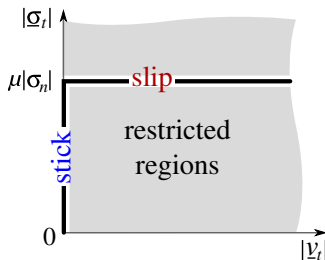
On slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

- **Complementary condition**

Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

$$|\underline{v}_t| \left(|\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$



Improved scheme explaining frictional contact conditions

Amontons-Coulomb's friction

- **No contact** $g > 0$, $\sigma_n = 0$

- **Stick** $|\underline{v}_t| = 0$

Inside slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip** $|\underline{v}_t| > 0$

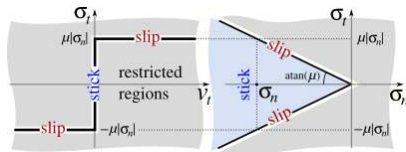
On slip surface/Coulomb's cone

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| = 0$$

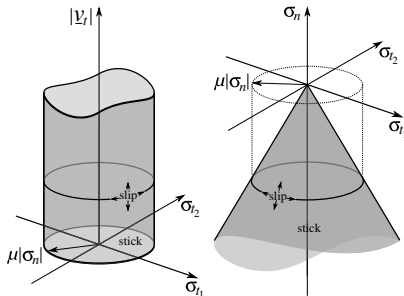
- **Complementary condition**

Either zero velocity and negative slip criterion, or non-zero velocity and zero slip criterion

$$|\underline{v}_t| \left(|\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$



Scheme of 2D frictional contact

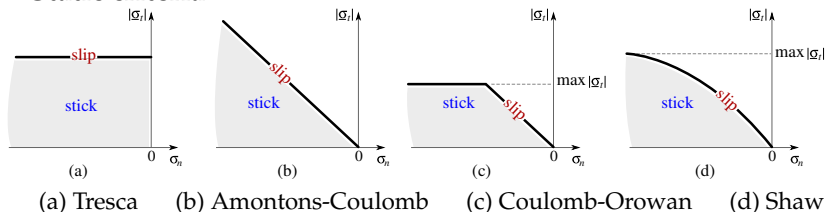


Scheme of 3D frictional contact

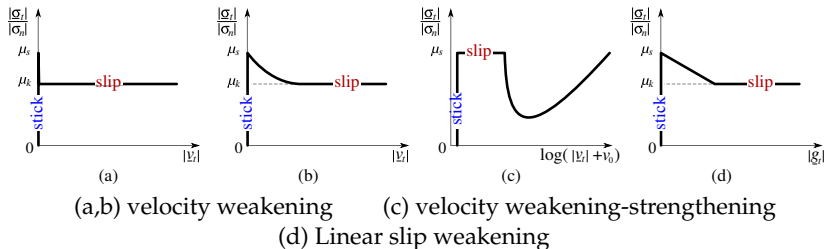
$$|\underline{v}_t| \geq 0, \quad |\underline{\sigma}_t| - \mu|\sigma_n| \leq 0, \quad |\underline{v}_t| \left(|\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$

More friction laws

• Static criteria



• Kinetic criteria



- μ_s static and μ_k kinetic coefficients of friction.

Rate and state friction and regularization

- **Rate and state friction law**

- Rate $v_t = |\underline{v}_t|$ – relative slip velocity
- State $\theta \approx$ internal time
- Dieterich–Ruina–Perrin (1979, 83, 95)

Frictional resistance

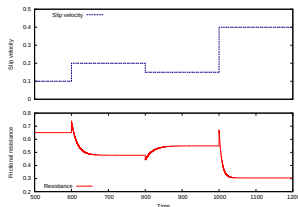
$$\sigma_t^c = |\sigma_n| [\mu_s + b\theta + a \ln(v_t/v_0)]$$

Evolution of the state variable

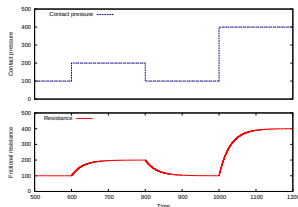
$$\dot{\theta} = -\frac{v_t}{L} \left[\theta + \ln\left(\frac{v_t}{v_0}\right) \right]$$

- **Prakash-Clifton friction law (1992,2000)**

- Viscous type evolution of frictional resistance σ_t
- $\dot{\sigma}_t = -\frac{v_t}{L} (\sigma_t + \mu \sigma_n)$



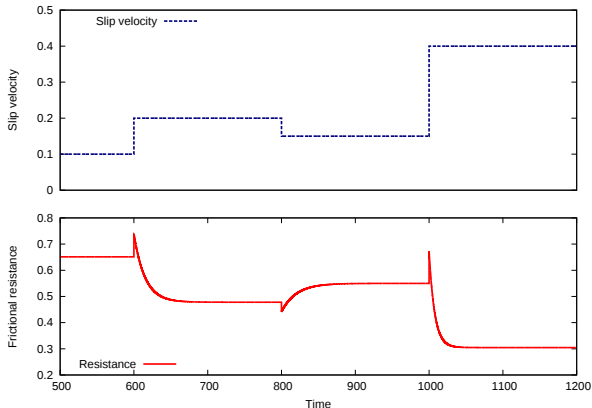
Rate and state friction law



Prakash-Clifton regularization

Rate and state friction and regularization

- Rate and state friction law



Rate and state friction and regularization

- **Rate and state friction law**

- Rate $v_t = |\underline{v}_t|$ – relative slip velocity
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Frictional resistance

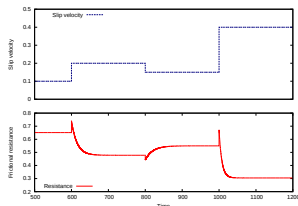
$$\sigma_t^c = |\sigma_n| [\mu_s + b\theta + a \ln(v_t/v_0)]$$

Evolution of the state variable

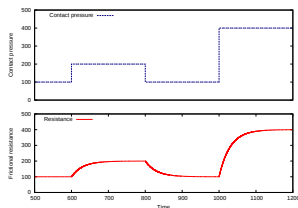
$$\dot{\theta} = -\frac{v_t}{L} \left[\theta + \ln\left(\frac{v_t}{v_0}\right) \right]$$

- **Prakash-Clifton friction law (1992,2000)**

- Viscous type evolution of frictional resistance σ_t
- $\dot{\sigma}_t = -\frac{v_t}{L} (\sigma_t + \mu \sigma_n)$



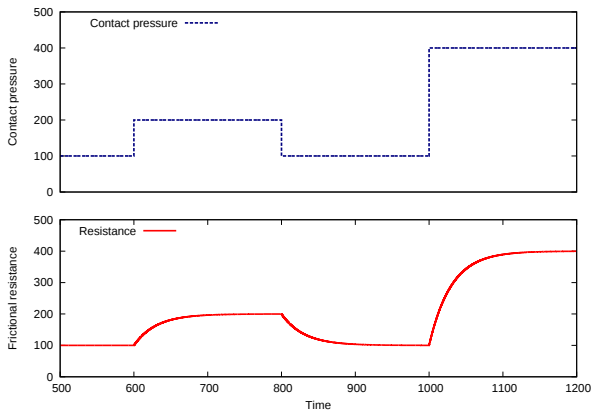
Rate and state friction law



Prakash-Clifton regularization

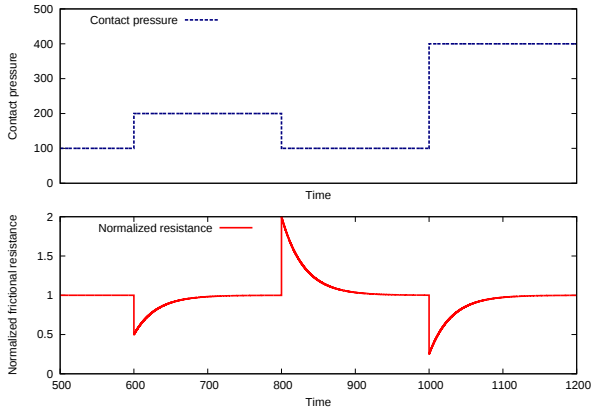
Rate and state friction and regularization

- Prakash-Clifton friction law (1992,2000)



Rate and state friction and regularization

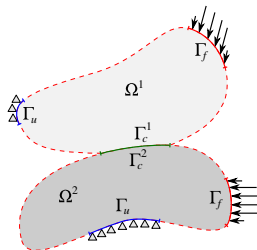
- Prakash-Clifton friction law (1992,2000)



From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$



Two solids in contact

From strong to weak form

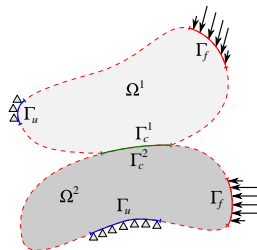
- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works



$$\boxed{\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma} + \int_{\Omega} [\underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} - \underline{\underline{\sigma}} : \delta \nabla \underline{\underline{u}}] d\Omega = 0$$



Two solids in contact

From strong to weak form

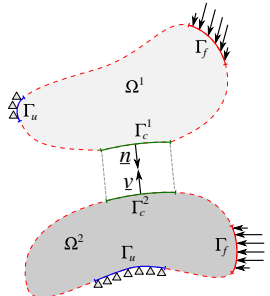
- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + f_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works

$$\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{\delta u} d\Gamma =$$

$$\int_{\bar{\Gamma}_c^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\rho} d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_c^2} \underline{v} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{r} d\bar{\Gamma}_c^2 + \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma_f$$



Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

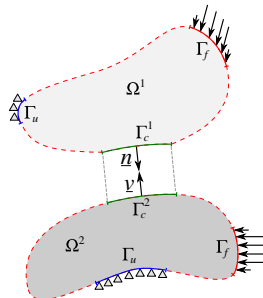
- Balance of virtual works



$$\int_{\partial\Omega} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{u} \, d\Gamma \Rightarrow$$

$$\int_{\bar{\Gamma}_\ell^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\rho} \, d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_\ell^2} \underline{v} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{r} \, d\bar{\Gamma}_c^2 =$$

$$= \int_{\bar{\Gamma}_\ell^1} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \delta (\underline{\rho} - \underline{r}) \, d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_\ell^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1$$



Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works



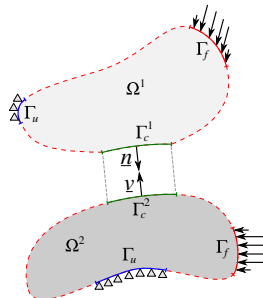
$$\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma \Rightarrow$$

$$\int_{\bar{\Gamma}_c^1} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{\rho}} d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_c^2} \underline{\underline{v}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{r}} d\bar{\Gamma}_c^2 =$$

$$= \int_{\bar{\Gamma}_c^1} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\underline{\rho}} - \underline{\underline{r}}) d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1$$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \underbrace{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega$$

Contact term



Two solids in contact

From strong to weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works



$$\underbrace{\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega}_{\text{Change of the internal energy}} + \underbrace{\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} =$$

Change of the internal energy

Contact term

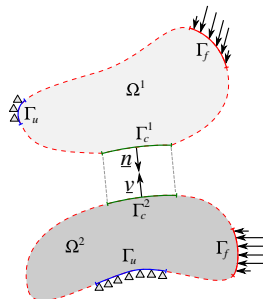
$$\underbrace{\int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega}_{\text{Virtual work of volume forces}}$$

Virtual work of external forces

Virtual work of volume forces

- Functional space

$\underline{\underline{u}} \in \mathbb{H}^1(\Omega)$ Sobolev space of the first order
and $\underline{\underline{u}}$ satisfy boundary and **contact conditions**.



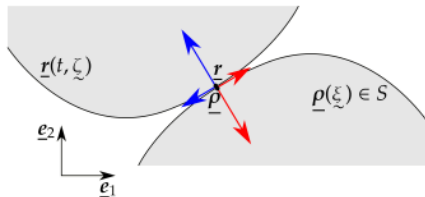
Two solids in contact

Towards variational inequality

■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\varrho}_t^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Contact configuration $\sigma_n \delta g_n = 0$, $\sigma_n \leq 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\varrho}_t^T \delta \underline{\xi} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

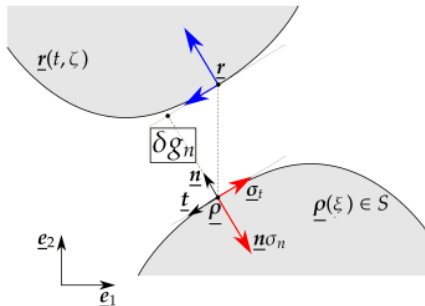
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\varrho}_t^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Virtual change of the configuration

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\varrho}_t^T \delta \underline{\xi} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

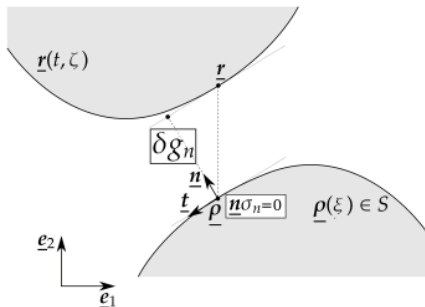
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\varrho}_t^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Normal term in separation $\delta g_n > 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\varrho}_t^T \delta \underline{\xi} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

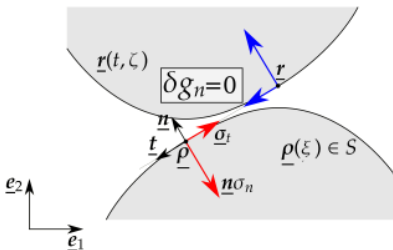
$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Towards variational inequality

■ Contact term

$$\int_{\bar{\Gamma}_c^1} \left(\sigma_n \delta g_n + \underline{\sigma}_t^T \delta \underline{\xi} \right) d\bar{\Gamma}_c^1$$

$$\int_{\bar{\Gamma}_c^1} \sigma_n \delta g_n d\bar{\Gamma}_c^1 \leq 0$$



Normal term in sliding $\delta g_n = 0$

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\sigma}_t^T \delta \underline{\xi} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u, g_n(\underline{u} + \delta \underline{u}) \geq 0 \text{ on } \Gamma_c \right\}$$

Back to variational equality (unconstrained)

- Constrained minimization problem

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\underline{\sigma}}_i^T \delta \underline{\underline{\xi}} d\bar{\Gamma}_c^1 \geq \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u, g_n(\underline{\underline{u}} + \delta \underline{\underline{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

- Use optimization theory to convert to

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\Gamma_c^1} \underbrace{C(\sigma_n, \sigma_t, g_n, \underline{\underline{\xi}}, \delta \underline{\underline{u}})}_{\text{Contact term}^*} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

Unconstrained functional space $\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u \right\}$

Contact term* is defined on the *potential contact zone* Γ_c^1 .

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

■ Penalty method

- New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2} = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 \quad \text{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 \quad \text{contact} \end{cases}$$

where ϵ is the penalty parameter.

- Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \langle -g(\mathbf{x}) \rangle \nabla g(\mathbf{x}) = 0$$

- Solution **tends** to the precise solution as $\epsilon \rightarrow \infty$

■ Lagrange multipliers method

■ Augmented Lagrangian method

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

■ Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

■ **Lagrange multipliers method**

- New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$$

- Saddle point problem

$$\min_{\mathbf{x}} \max_{\lambda} \{\mathcal{L}(\mathbf{x}, \lambda)\} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x}) \geq 0} \{F(\mathbf{x})\}$$

- Stationary point

$$\nabla_{\mathbf{x}, \lambda} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0 \quad \text{need to verify} \quad \lambda \leq 0$$

■ **Augmented Lagrangian method**

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Optimization methods: recall

Functional to be minimized $F(\mathbf{x})$ under constraint $g(\mathbf{x}) \geq 0$

■ Penalty method $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

■ Lagrange multipliers method $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$

■ **Augmented Lagrangian method**

[Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]

• New functional, augmented Lagrangian

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \lambda g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \text{ non-contact} \end{cases}$$

• Stationary point

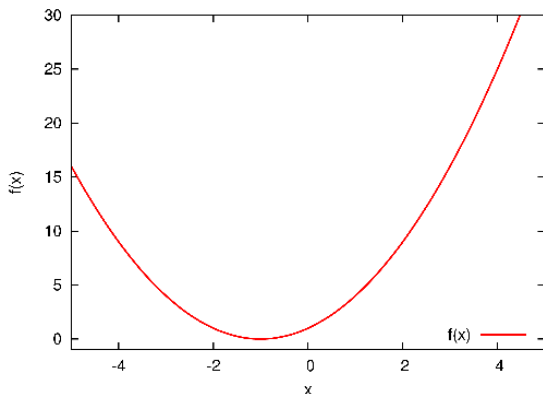
$$\nabla_{\mathbf{x}, \lambda} \mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \text{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \text{if non-contact} \end{cases}$$

Macaulay brackets $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$



Uzawa algorithm

Optimization methods: example

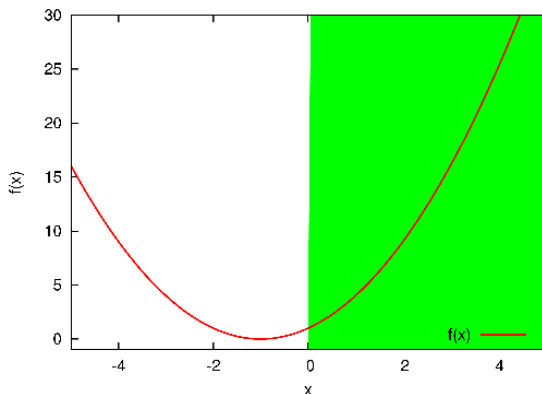


Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

Solution : $x^* = 0$

Optimization methods: example



Functional : $f(x) = x^2 + 2x + 1$

Constrain : $g(x) = x \geq 0$

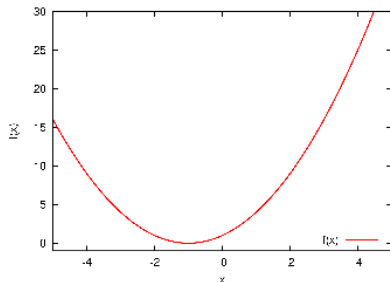
Solution : $x^* = 0$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$

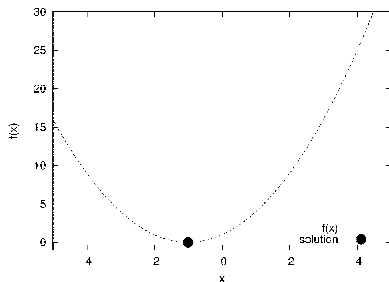
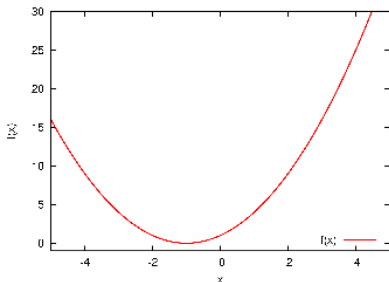


Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



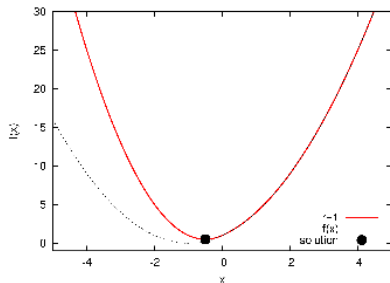
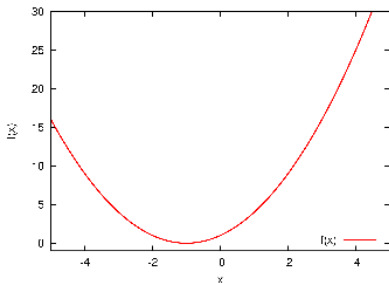
$\epsilon = 0$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



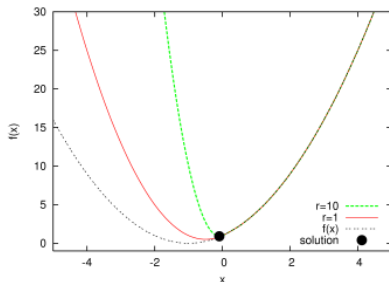
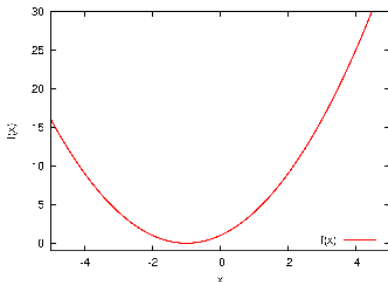
$\epsilon = 1$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



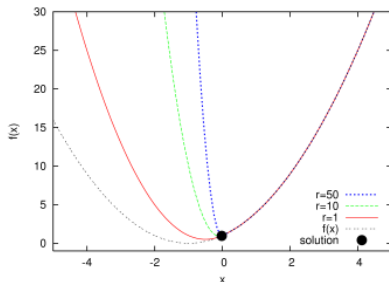
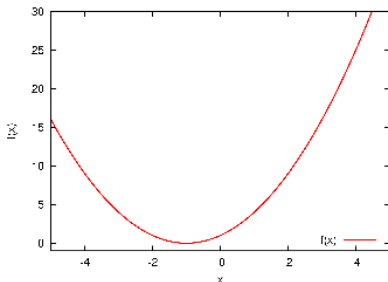
$$\epsilon = 10$$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$



$\epsilon = 50$

Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$

Advantages 😊

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- “mathematically” smooth functional

Drawbacks 😞

- practically non-smooth functional
- solution is not exact:
 - too small penalty $\rightarrow$ large penetration
 - too large penalty $\rightarrow$ ill-conditioning of the tangent matrix
- user has to choose penalty ϵ properly or automatically and/or adapt during convergence

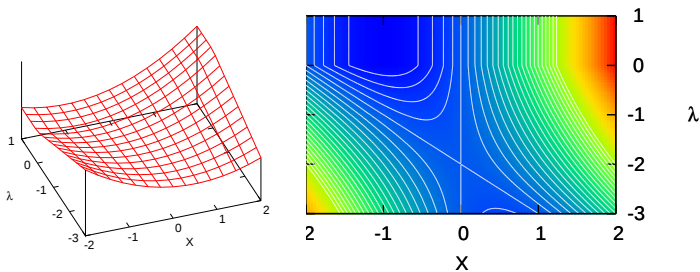
Lagrange multipliers method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Lagrange multipliers method

$$\mathcal{L}(x, \lambda) = F(x) + \boxed{\lambda g(x)} \rightarrow \text{Saddle point} \rightarrow \min_x \max_{\lambda} \mathcal{L}(x, \lambda)$$

Need to check that $\lambda \leq 0$



Lagrange multipliers method: example

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Need to check that $\lambda \leq 0$

Advantages 😊

- exact solution
- no adjustable parameters

Drawbacks 😞

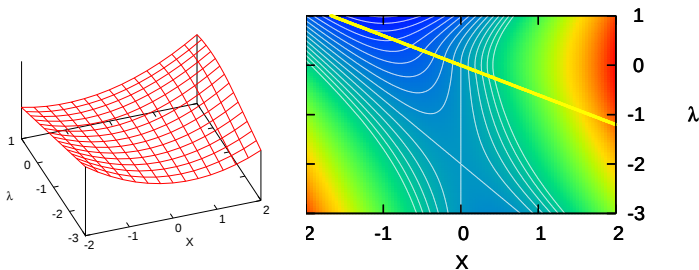
- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained: $\lambda \leq 0$

Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

■ Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \boxed{\lambda g(\mathbf{x})} + \boxed{\epsilon g^2(\mathbf{x})}, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \geq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) < 0, \text{ non-contact} \end{cases}$$



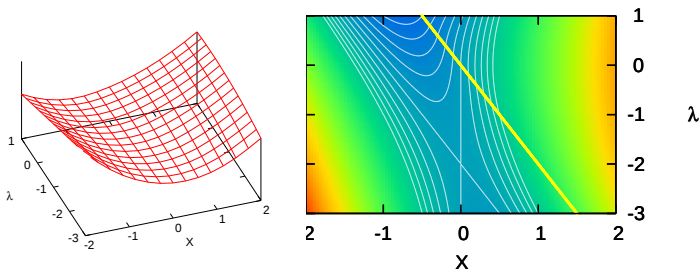
Yellow line separates contact and non-contact regions

Augmented Lagrangian method: example

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■ Augmented Lagrangian method

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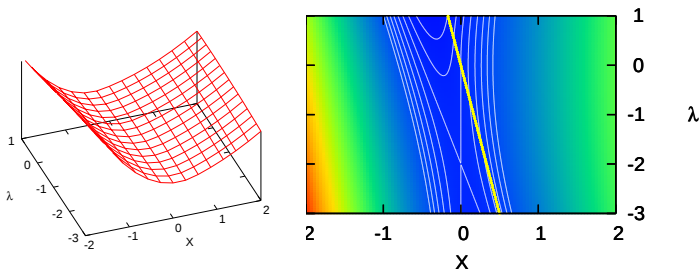
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Augmented Lagrangian method: example

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Advantages 😊

- exact solution
- smooth functional (!)
- fully unconstrained

Drawbacks 😞

- additional degrees of freedom
- quite sensitive to parameter ϵ
- need to adjust ϵ during convergence

Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} d\Omega + \underbrace{\int_{\Gamma_c^1} \boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u \right\}$$

■ Penalty method

$$\text{Pressure: } \sigma_n = \epsilon g_n, \text{ Shear: } \underline{\underline{\sigma}}_t = \begin{cases} \epsilon \underline{g}_t, & \text{if stick } |\sigma_t| < \mu |\sigma_n| \\ \mu \epsilon g_n \delta \underline{g}_t / |\delta \underline{g}_t|, & \text{if slip } |\sigma_t| = \mu |\sigma_n| \end{cases}$$

Contact term

$$C = C(g_n, \underline{g}_t, \delta g_n, \delta \underline{g}_t) = \sigma_n \delta g_n + \underline{\underline{\sigma}}_t \cdot \delta \underline{g}_t$$

Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \cdot \delta \nabla \underline{u} d\Omega + \underbrace{\int_{\Gamma_c^1} \boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\sigma}_0 \cdot \delta \underline{u} d\Gamma + \int_{\Omega} \underline{f}_v \cdot \delta \underline{u} d\Omega,$$

$$\mathbb{K} = \left\{ \delta \underline{u} \in \mathbb{H}^1(\Omega) \mid \delta \underline{u} = 0 \text{ on } \Gamma_u \right\}$$

■ Augmented Lagrangian method

Contact term

$$C = C(g_n, \underline{g}_t, \lambda_n, \underline{\lambda}_t, \delta g_n, \delta \underline{g}_t, \delta \lambda_n, \delta \underline{\lambda}_t)$$

$$C = \begin{cases} -\frac{1}{\epsilon} \left(\lambda_n \delta \lambda_n - \underline{\lambda}_t \cdot \delta \underline{\lambda}_t \right), & \text{if non-contact } \lambda_n + \epsilon g_n \geq 0 \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \underline{\hat{\lambda}}_t \cdot \delta \underline{g}_t + \underline{g}_t \cdot \delta \underline{\hat{\lambda}}_t, & \text{if stick } |\underline{\hat{\lambda}}_t| \leq \mu |\hat{\sigma}_n| \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \mu \hat{\sigma}_n - \mu \hat{\sigma}_n \frac{\hat{\lambda}_t}{|\underline{\hat{\lambda}}_t|} \cdot \delta \underline{g}_t - \frac{1}{\epsilon} \left(\lambda_t + \mu \hat{\sigma}_n \frac{\hat{\lambda}_t}{|\underline{\hat{\lambda}}_t|} \right) \cdot \delta \underline{\lambda}_t, & \text{if slip } |\underline{\hat{\lambda}}_t| \geq \mu |\hat{\sigma}_n| \end{cases}$$

where $\hat{\lambda}_n = \lambda_n + \epsilon g_n$ and $\underline{\hat{\lambda}}_t = \underline{\lambda}_t + \epsilon \underline{g}_t$.



The scream

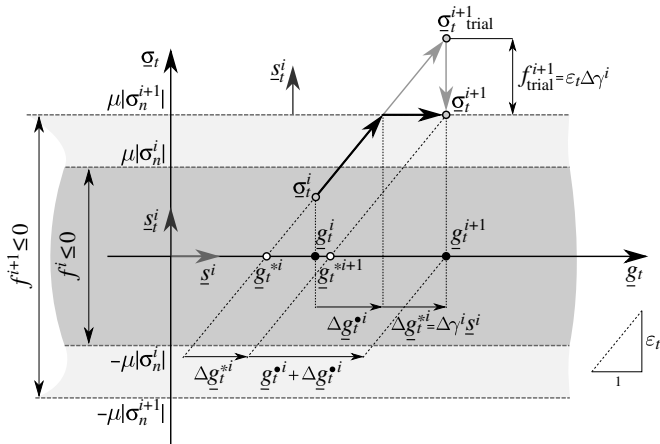
Friction: methods

- Optimization methods: penalty or augmented Lagrangian method
- Return mapping algorithm for penalty
- Analogy with elasto-plastic formulation problem^[1]

[1] Curnier "A theory of friction" Int J Solids Struct 20 (1984)

Friction: Return mapping algorithm

- Return mapping algorithm in 2D

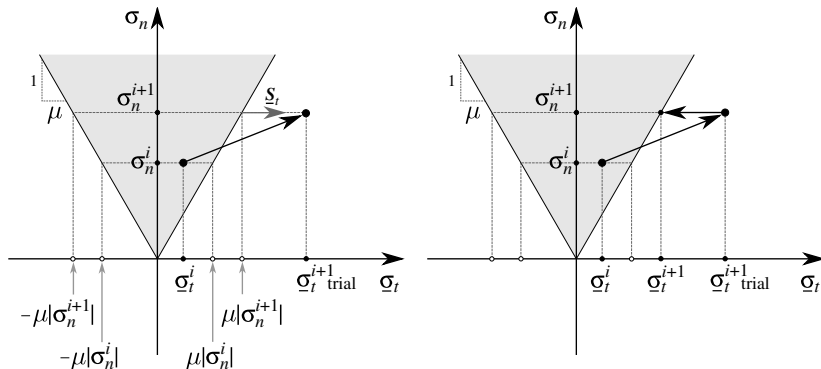


As in plasticity^[1]

[1] Simo J.C. and Hughes T.J.. Computational inelasticity. Springer (2006)

Friction: Return mapping algorithm

■ Return mapping algorithm in 2D

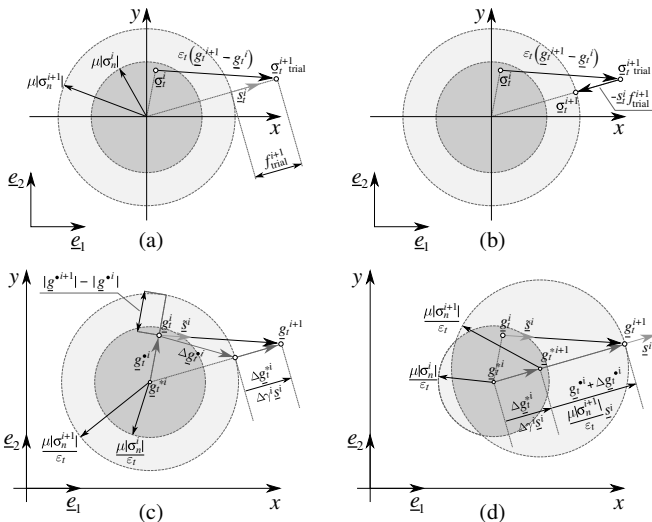


Analogy with non-associated plastic flow^[2]

[2] Curnier A. A theory of friction. International Journal of Solids and Structures 20 (1984)

Friction: Return mapping algorithm

Return mapping algorithm in 3D



A slightly more messy thing

Application to contact problems: linearization

- Non-linear equation

$$R(\underline{u}, \underline{f}) = 0$$

- Contains $\delta g_n, \delta \underline{g}_t$
- Use Newton-Raphson method
- Initial state at step i

$$R(\underline{u}^i, \underline{f}^i) = 0$$

- Should be also satisfied at step $i + 1$

$$R(\underline{u}^{i+1}, \underline{f}^{i+1}) = R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = 0$$

- Linearize

$$R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = R(\underline{u}^i, \underline{f}^{i+1}) + \frac{\partial R(\underline{u})}{\partial \underline{u}} \delta \underline{u} = 0$$

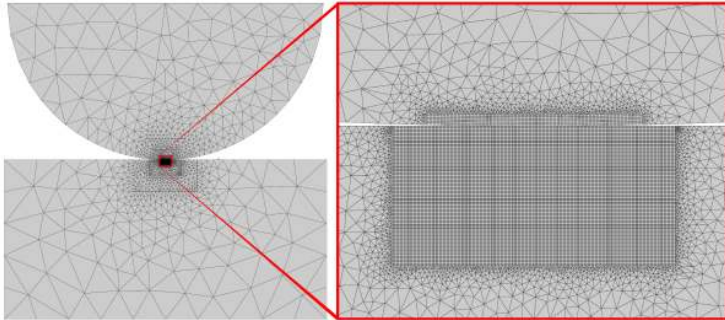
- Finally

$$\delta \underline{u} = - \underbrace{\left[\frac{\partial R(\underline{u})}{\partial \underline{u}} \right]^{-1}}_{\text{contains } \Delta \delta g_n, \Delta \delta \underline{g}_t} R(\underline{u}^i)$$

- NB: Contact problem does not satisfy conditions of Kantorovich theorem on the convergence of Newton's method.

Particularities: mesh and convergence

- Strong **mesh refinement** is required
 - especially at **unknown edges** of contact zones

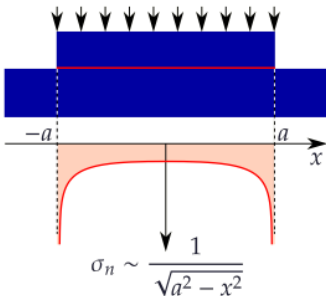


Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]

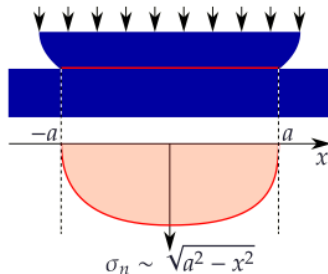
2D ~ 30 000 DoFs, 3D ~ 5 000 000 DoFs

Particularities: mesh and convergence

- Strong **mesh refinement** is required
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$$\sigma_n \xrightarrow{x \rightarrow a} -\infty \quad \left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$



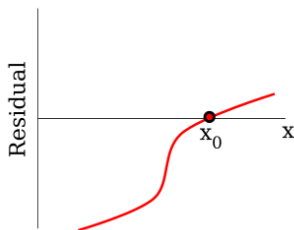
$$\left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$

Infinite contact pressure and/or its derivative

Particularities: mesh and convergence

- Strong **mesh refinement** is required
 - especially at **unknown edges** of contact zones
- **Slow change** of boundary conditions:
 - strong non-linearities of contact/friction problems
 - non-uniqueness of solution for frictional problems

Infinite looping

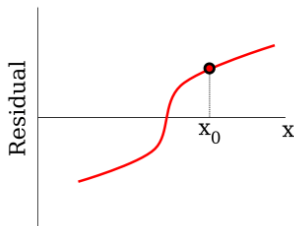


Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

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Infinite looping

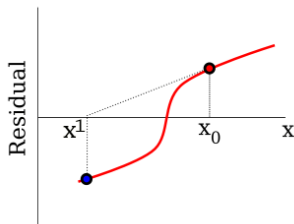


Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

Particularities: mesh and convergence

- Strong **mesh refinement** is required
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 - strong non-linearities of contact/friction problems
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Infinite looping



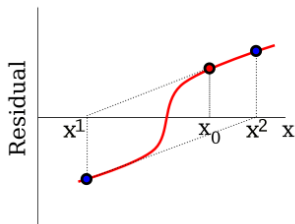
Iterations of Newton-Raphson method

$$R(x_0, f_1) + \left. \frac{\partial R}{\partial x} \right|_{x_0} \delta x = 0 \rightarrow \delta x = - \left. \frac{\partial R}{\partial x} \right|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

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Infinite looping



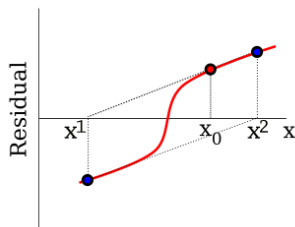
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Particularities: mesh and convergence

- Strong **mesh refinement** is required
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Infinite looping

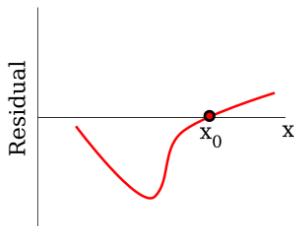


Infinite looping

Particularities: mesh and convergence

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Convergence to a “false” solution

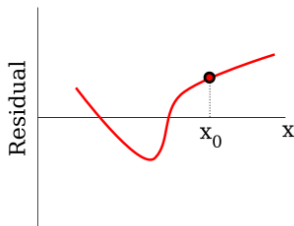


Initial guess $R(x_0, f_0) = 0$

Particularities: mesh and convergence

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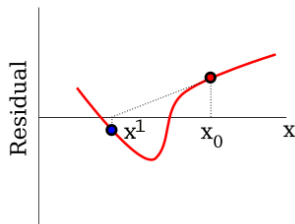


Too rapid change in boundary conditions $R(x_0, f_1) \neq 0$

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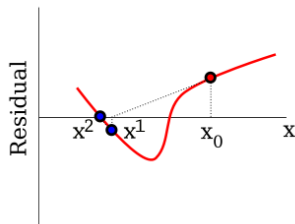
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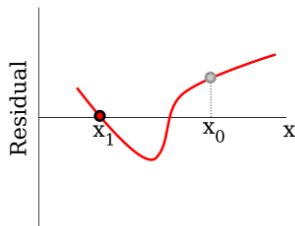
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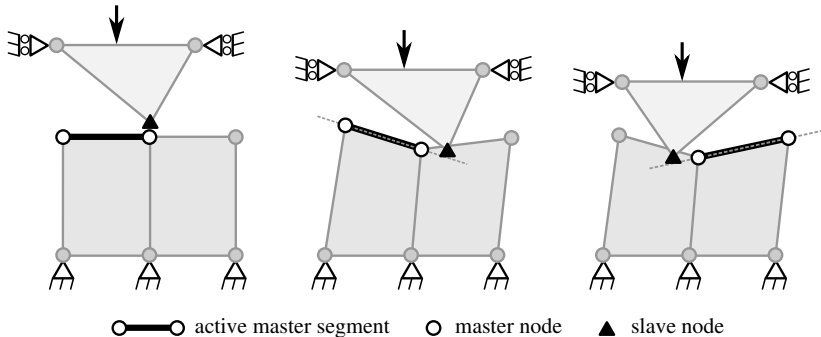
Convergence to a “false” solution



Convergence, but is it a “true” solution ?

Convergence problems: examples

■ Infinite looping, e.g.

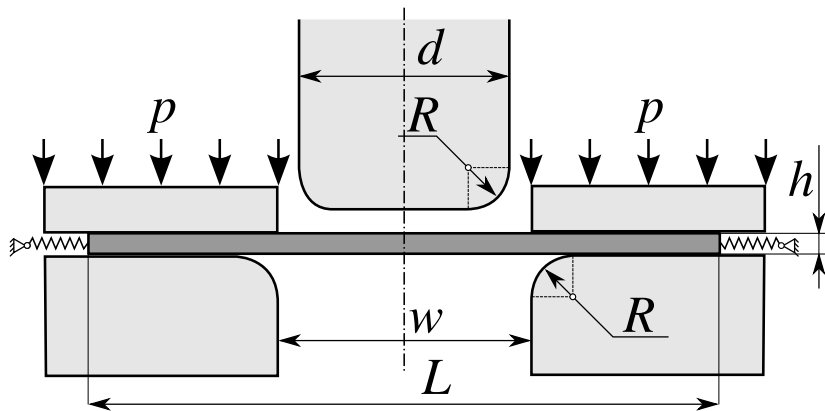


- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients^[1]
- Combination of non-linearities (e.g., plasticity+contact)

Alart P, Méthode de Newton généralisée en mécanique du contact Journal de Mathématiques Pures et Appliquées 76 (1997)

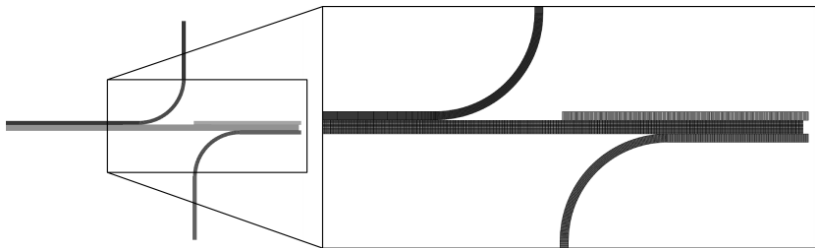
Convergence problems: examples

- Simulation of a deep drawing problem
- Finite strain plasticity + frictional contact



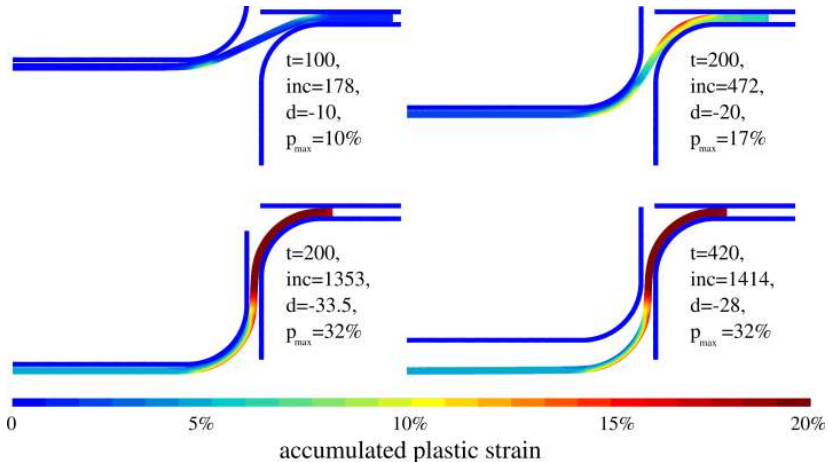
Convergence problems: examples

- Simulation of a deep drawing problem
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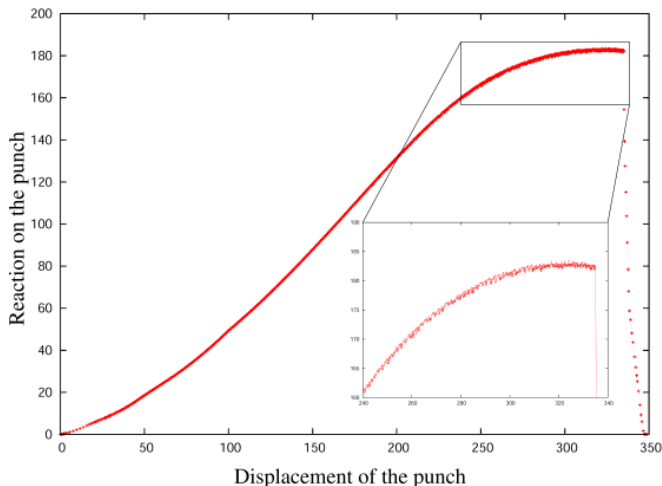
Convergence problems: examples

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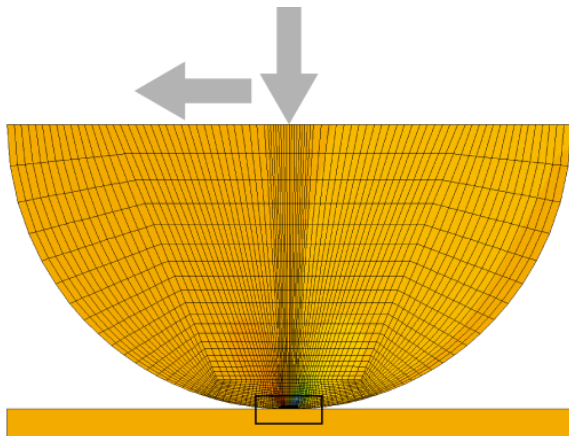
Convergence problems: examples

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Cylinder-plane frictional contact

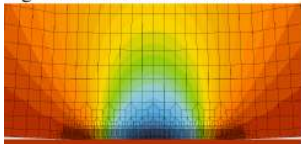
- Non-conservative problem, history of loading is crucial



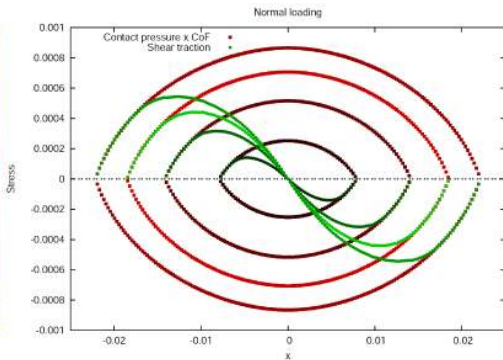
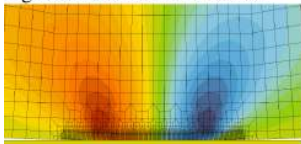
Cylinder-plane frictional contact

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sig22 at maximal normal load



sig12 at maximal normal load

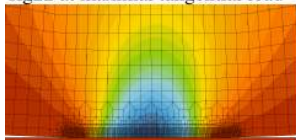


Press in 100 increments, $u_z \sim t^2$

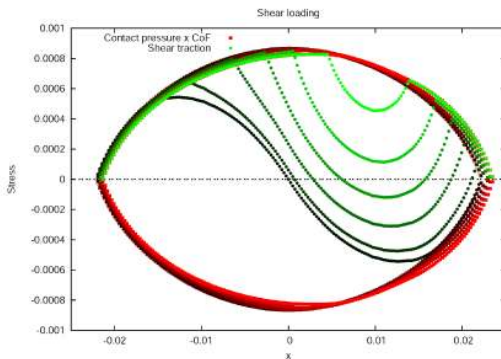
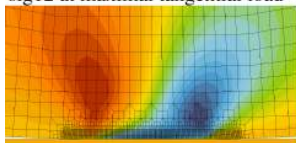
Cylinder-plane frictional contact

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sig22 at maximal tangential load



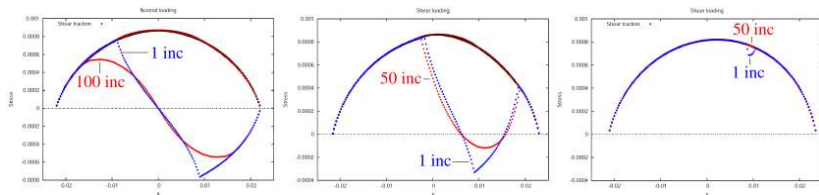
sig12 at maximal tangential load



Shift in 100 increments, $u_z \sim t$

Cylinder-plane frictional contact

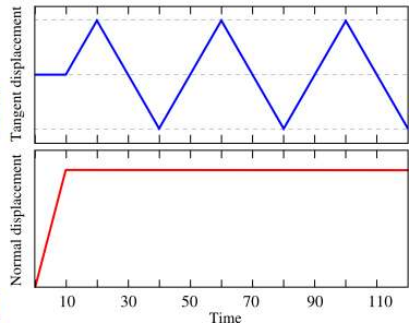
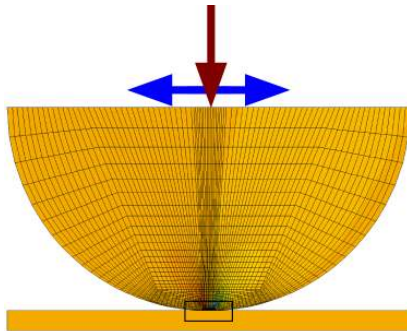
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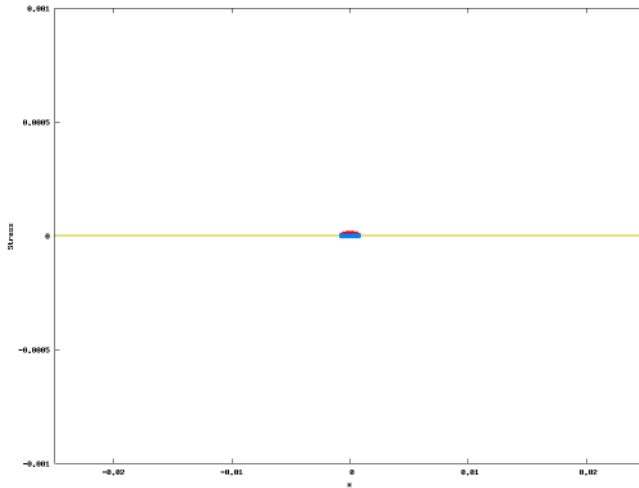
Comparison with: press in 1 increment, shift in 2 increments

Before stick every point of the contact interface has to pass through the slip zone. It is impossible when loaded too fast.

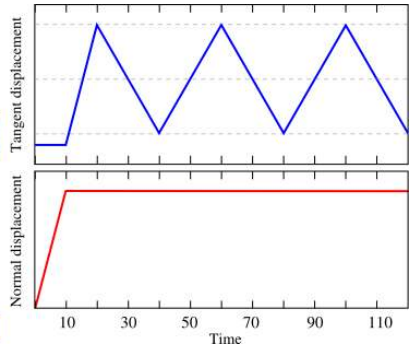
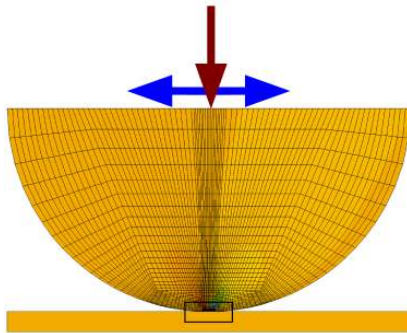
Sphere-plane frictional contact: cycling



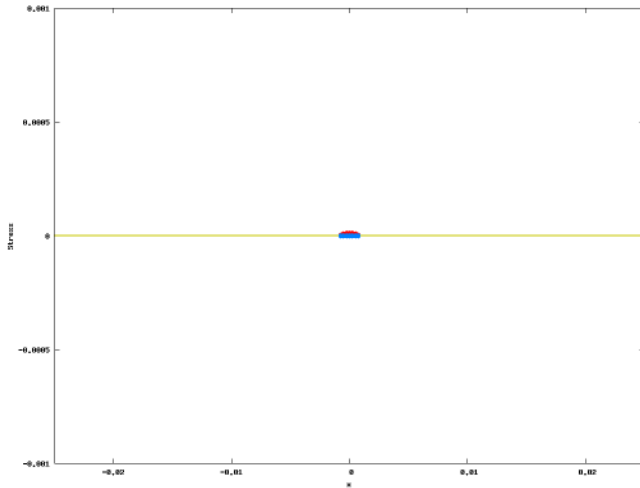
Sphere-plane frictional contact: cycling



Sphere-plane frictional contact: cycling

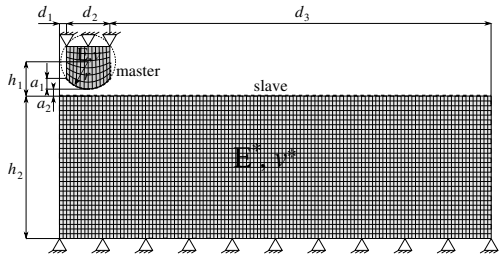


Sphere-plane frictional contact: cycling



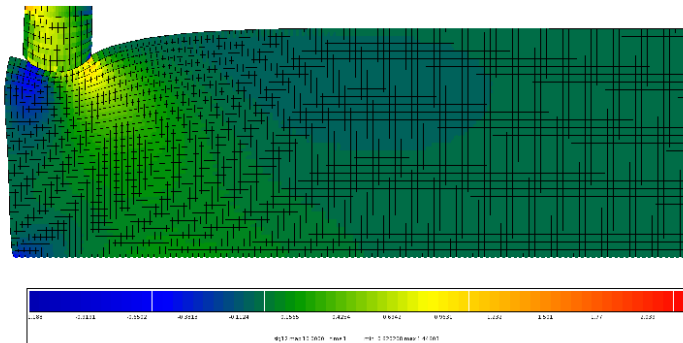
Shallow ironing test

- Deformable-on-deformable frictional sliding
- Results obtained by different groups^{1,2,3,4,5,6} differ significantly
- Local and global friction coefficients may differ



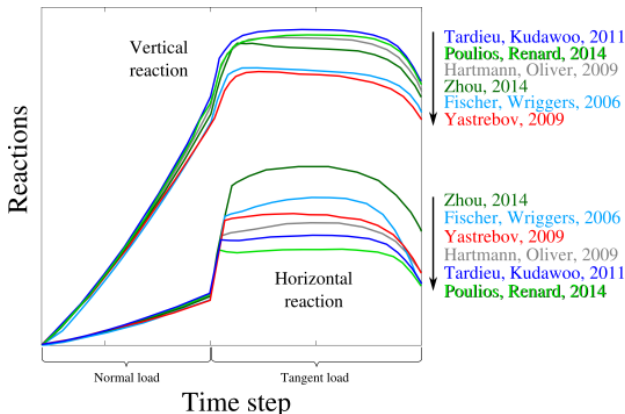
- [1] Fischer K. A., Wriggers P., "Mortar based frictional contact formulation for higher order interpolations using the moving friction cone", Computer Methods in Applied Mechanics and Engineering, vol. 195, p. 5020-5036, 2006.
- [2] Hartmann S., Oliver J., Cante J. C., Weyler R., Hernández J. A., "A contact domain method for large deformation frictional contact problems. Part 2: Numerical aspects", Computer Methods in Applied Mechanics and Engineering, vol. 198, p. 2607-2631, 2009.
- [3] Yastrebov V. A., "Computational contact mechanics: geometry, detection and numerical techniques", Thèse CdM & Onera, 2011.
- [4] Kudawoo A. D., "Problèmes industriels de grande dimension en mécanique numérique du contact : performance, fiabilité et robustesse", Thèse @ LMA & LAMSID, 2012.
- [5] Poulis K., Renard Y., "A non-symmetric integral approximation of large sliding frictional contact problems of deformable bodies based on ray-tracing", soumis, 2014.
- [6] Zhou Lei's blog, <http://kt2008plus.blogspot.de>

Shallow ironing test



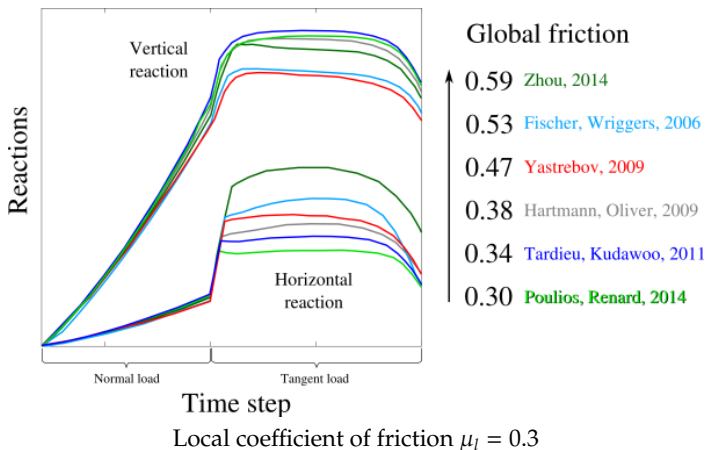
Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



Shallow ironing test

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Examples of contact problems

With analytical solution

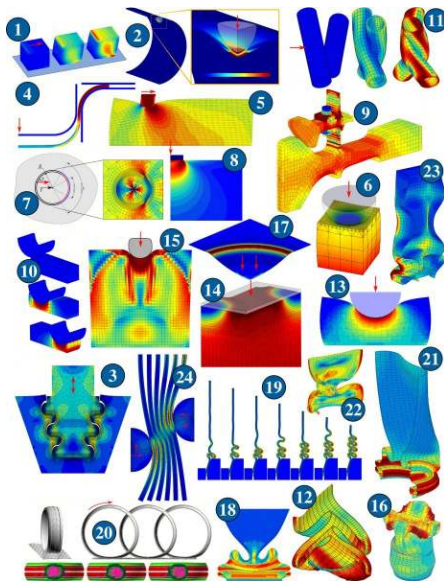
- ★ linear elasticity
- ★ with/without friction

From literature

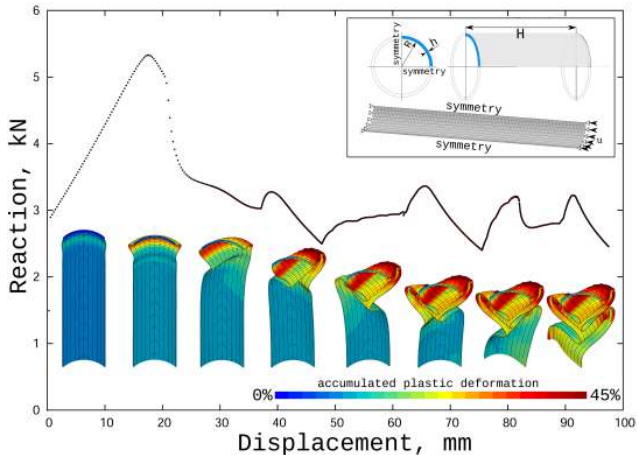
- ★ post-buckling 2D
- ★ finite strains
- ★ elasticity / plasticity
- ★ with/without friction

New

- ★ multi-contacts
- ★ post-buckling 3D
- ★ finite strains
- ★ elasticity / plasticity
- ★ with/without friction



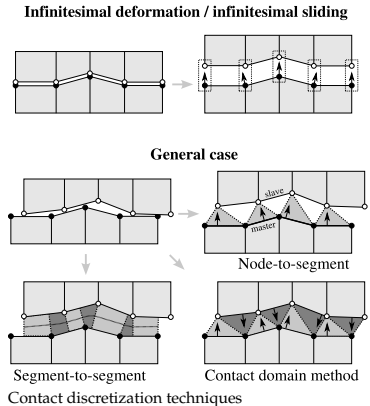
Self-contact problem



Finite element analysis of a post-buckling behavior of a thin walled tube
Collection of non-linearities: buckling instability, self-contact, finite strain plasticity

- It's just a tip of the “Computational Contact Mechanics” iceberg
- Contact detection
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics

See Basava R. Akula presentation on Friday



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Kikuchi, Oden (1988)



Zhong (1993)



Wriggers (2002)



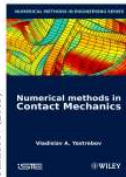
Wriggers, 2nd ed. (2006)



Laursen (2002)



Yastrebov (2013)



- It's just a tip of the “Computational Contact Mechanics” iceberg
- Contact detection
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- Several advanced topics

Kikuchi, Oden (1988)



Wriggers (2002)



Laursen (2002)



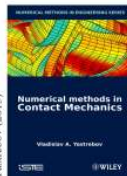
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$\mathcal{L}_a(x, \lambda)$

Thank you for your attention!
