Finite Element Method for Continuum Solid Mechanics

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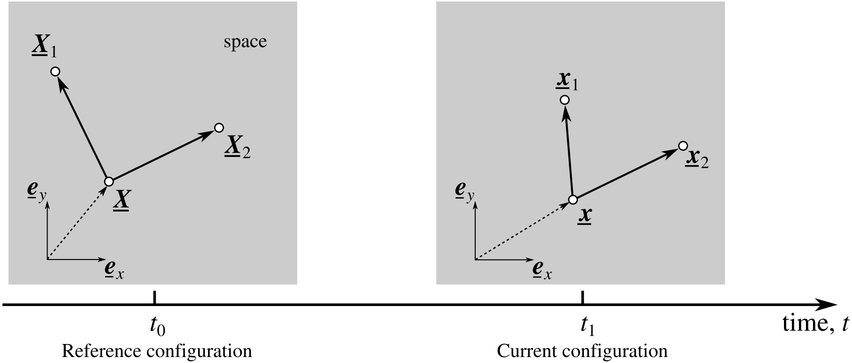
Outline

- Reminder: Continuum Solid Mechanics
- Finite Element Method
- Mesh adaptivity and convergence
- Examples

Continuum Solid Mechanics: a Reminder

Deformable medium

- Deformation in time t
- Reference configuration at $t = t_0$, $\underline{X}$ and current configuration at $t = t_1$, $\underline{x}(\underline{X}, t)$
- Lagrangian description, follow material points $\underline{X} = \underline{x}(t = t_0)$
- Displacement vector is $\underline{u} = \underline{x} - \underline{X}$



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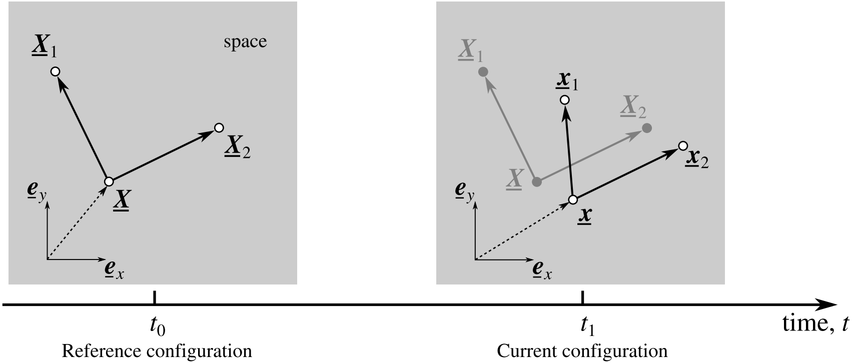

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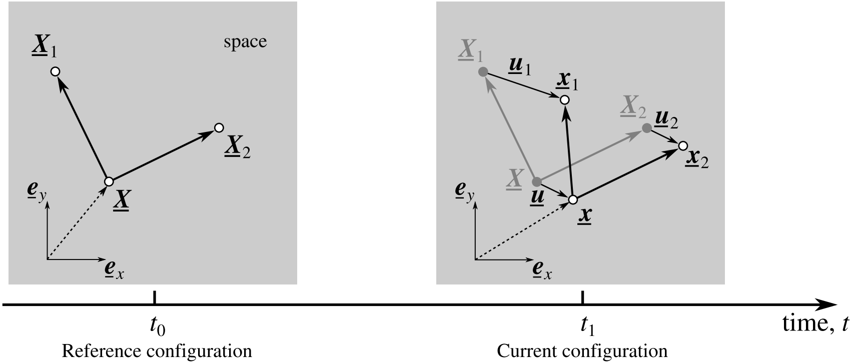
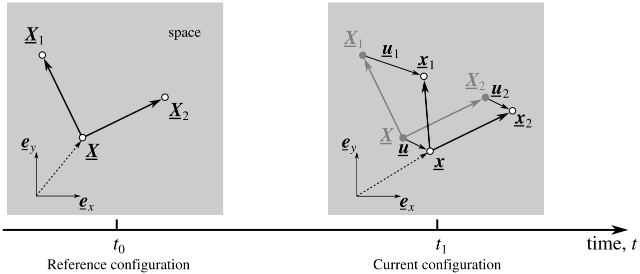


Deformation tensor

- Transformation gradient $\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{X}} = \frac{\partial\left(\underline{X} + \underline{u}\right)}{\partial \underline{X}} = \underline{\underline{I}} + \frac{\partial \underline{u}}{\partial \underline{X}} = \underline{\underline{I}} + \underline{\underline{H}}$
- Cauchy-Green right tensor $\underline{\underline{C}} = \underline{\underline{F}}^{\mathsf{T}} \cdot \underline{\underline{F}}$
- Green-Lagrange deformation tensor $\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{C}} - \underline{\underline{I}}\right) = \underline{\underline{H}}^{S} + \frac{1}{2}\underline{\underline{H}}^{\mathsf{T}} \cdot \underline{\underline{H}}$
- For $H_{ij} \ll 1$, $\underline{\underline{E}} \approx \underline{\underline{H}}^{S}$ and we obtain a tensor of small deformations

$$\underline{\underline{\varepsilon}} = \underline{\underline{H}}^{S} = \frac{1}{2}\left[\frac{\partial \underline{u}}{\partial \underline{X}} + \left(\frac{\partial \underline{u}}{\partial \underline{X}}\right)^{\mathsf{T}}\right] = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^{\mathsf{T}}\right)$$



Stress tensor and Hooke's law

- Hooke's law in uniaxial test:

$$\boxed{\sigma_{xx} = E\varepsilon_{xx}}$$

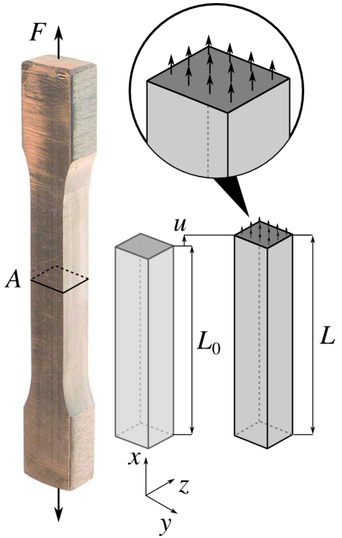
$$F = ku \quad \Leftrightarrow \quad \sigma_{xx}A = \frac{EA}{L_0}u = EA\frac{L - L_0}{L_0}$$

- In general case stress and strain are related through a linear operator (fourth-order elasticity tensor ${}^4\underline{\underline{C}}$):

$$\boxed{\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : \underline{\underline{\varepsilon}}}$$

- Inversely the strain can be found through a stiffness tensor ${}^4\underline{\underline{S}}$:

$$\boxed{\underline{\underline{\varepsilon}} = {}^4\underline{\underline{S}} : \underline{\underline{\sigma}}}$$



Hooke's law for isotropic solids: stress

- In the case of isotropic material the Hooke's law reduces to:

$$\boxed{\underline{\underline{\sigma}} = \lambda \operatorname{tr}(\underline{\underline{\varepsilon}})\underline{\underline{I}} + 2\mu\underline{\underline{\varepsilon}},}$$

with λ, μ being Lamé coefficients:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

with Young's modulus E and Poisson's ratio ν.

- In the component form it reads:

$$\sigma_{ij} = \lambda(\varepsilon_{kk})\delta_{ij} + 2\mu\varepsilon_{ij}$$

- In the matrix form:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = 2\mu \begin{bmatrix} \lambda \operatorname{tr}(\underline{\varepsilon})/(2\mu) + \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \lambda \operatorname{tr}(\underline{\varepsilon})/(2\mu) + \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \lambda \operatorname{tr}(\underline{\varepsilon})/(2\mu) + \varepsilon_{33} \end{bmatrix}$$

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Hooke's law for isotropic solids: strain

- Strain as a function of stress:

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}}.$$

- In the component form it reads:

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$$= \frac{1}{E}\begin{bmatrix} \sigma_{11} - \nu(\sigma_{22}+\sigma_{33}) & (1+\nu)\sigma_{12} & (1+\nu)\sigma_{13} \\ (1+\nu)\sigma_{12} & \sigma_{22} - \nu(\sigma_{11}+\sigma_{33}) & (1+\nu)\sigma_{23} \\ (1+\nu)\sigma_{13} & (1+\nu)\sigma_{23} & \sigma_{33} - \nu(\sigma_{11}+\sigma_{22}) \end{bmatrix}$$

Equilibrium of an infinitesimal element

- Infinitesimal strain tensor is symmetric and satisfies the compatibility conditions*:

$$\nabla \times \left(\nabla \times \underline{\underline{\varepsilon}}\right) = 0$$

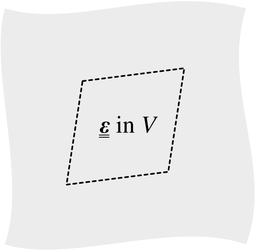
- Stress tensor $\underline{\underline{\sigma}}$ should ensure equilibrium of infinitesimal element**:

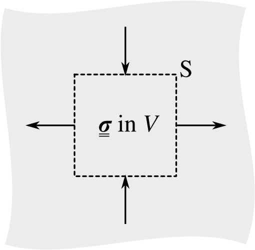
Force balance: $\int_S \underline{n} \cdot \underline{\underline{\sigma}}\, dS = 0$

Momentum balance: $\int_S \underline{r} \times (\underline{n} \cdot \underline{\underline{\sigma}})\, dS = 0$

- Following the divergence theorem:

$\int_S \underline{n} \cdot \underline{\underline{\sigma}}\, dS = \int_V \nabla \cdot \underline{\underline{\sigma}}\, dV = 0$ Since volume V can be arbitrary chosen, then
$\boxed{\nabla \cdot \underline{\underline{\sigma}} = 0}$ everywhere in V.





*In case of a simply-connected solid.
**In absence of volumetric forces.

Equilibrium of an infinitesimal element II

- Second Newton's law:

$$m\underline{\ddot{u}} = \underline{f} \quad \Rightarrow \quad \rho\underline{\ddot{u}} = \frac{1}{V}\underline{f}$$

- In presence of volumetric forces with density $\underline{f}_V$, the total force is given by:

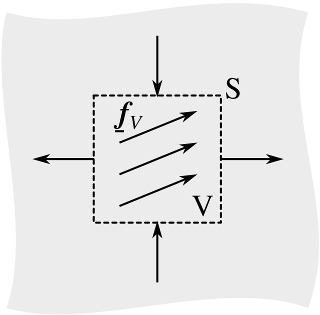
$$\underline{f} = \int_V \underline{f}_V \, dV + \int_S \underline{n} \cdot \underline{\underline{\sigma}} \, dS$$

- Then using the second Newton's law and the divergence theorem:

$$\int_V \left(\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V \right) dV = \int_V \rho\underline{\ddot{u}} \, dV$$

- Since it is right for arbitrary V, then in every point of V:

$$\boxed{\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V = \rho\underline{\ddot{u}}}$$



Equilibrium of an infinitesimal element II

- Equilibrium (3 equations):

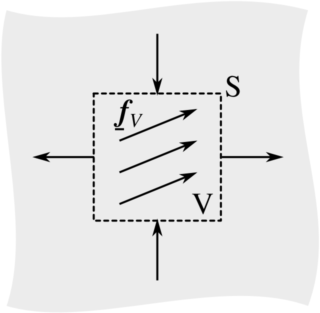
$$\nabla \cdot \underline{\underline{\boldsymbol{\sigma}}} + \underline{\boldsymbol{f}}_V = \rho \underline{\ddot{\boldsymbol{u}}}$$

- In component form*:

$$\sigma_{ij,j} + f_{Vi} = \rho \ddot{u}_i,$$

- Explicitly:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_{V_x} = \rho \ddot{u}_x$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_{V_y} = \rho \ddot{u}_y$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_{V_z} = \rho \ddot{u}_z$$



* The following notation is used $y_{i,j} = \frac{\partial y_i}{\partial x_j}$

Deformable solid and boundary conditions

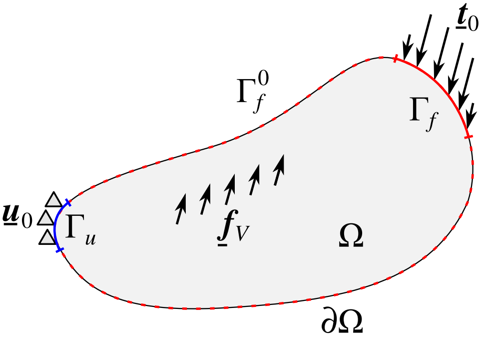
Notations:

- Consider a solid Ω with boundary $\partial\Omega$
- Boundary is split into Γ_u and Γ_f: $\partial\Omega = \Gamma_u \cup \Gamma_f$
- At Γ_u displacements $\underline{u}_0(t, \underline{X})$ are prescribed (Dirichlet boundary conditions [BC]):

  $\underline{u} = \underline{u}_0$ at Γ_u
- At Γ_f tractions $\underline{t}_0(t, \underline{X})$ are prescribed (Neumann BC):

  $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0$ at Γ_f

  $\underline{\underline{\sigma}} \cdot \underline{n} = 0$ at Γ_f^0



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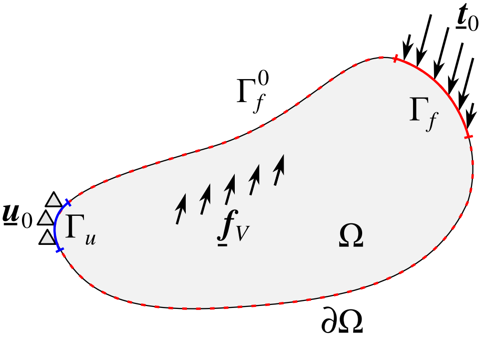


Remarks:

- on the same boundary both BCs can be prescribed if they are orthogonal one to each other, i.e. $\underline{u}_0 \cdot \underline{t}_0 = 0$ (*ex.:* friction);
- a relationship between these BCs can be prescribed (Robin BC): $\underline{u}_0 = \underline{U} - k\underline{t}_0$ (*ex.:* Winkler's foundation).

Elastic and static problem set-up

- Equilibrium in absence of inertial forces

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V = 0 \quad (*)$$

- Consistutive relation:

$$\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : \underline{\underline{\varepsilon}}$$

- Strain tensor:

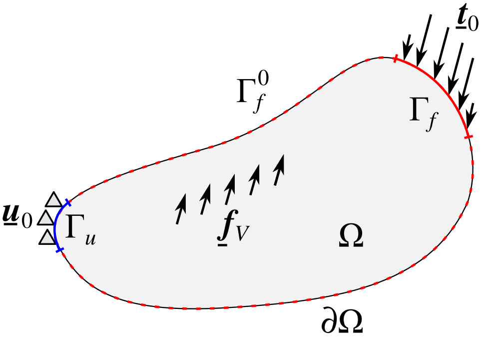
$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^{\top}\right)$$

- Boundary conditions:

$\underline{u} = \underline{u}_0$ at Γ_u

$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0$ at Γ_f

$\underline{\underline{\sigma}} \cdot \underline{n} = 0$ at Γ_f^0



- **Problem:**
find such field $\underline{u}$ in Ω that satisfies equilibrium Eq. $(*)$ and boundary conditions.

Finite Element Method

Main idea in a nutshell

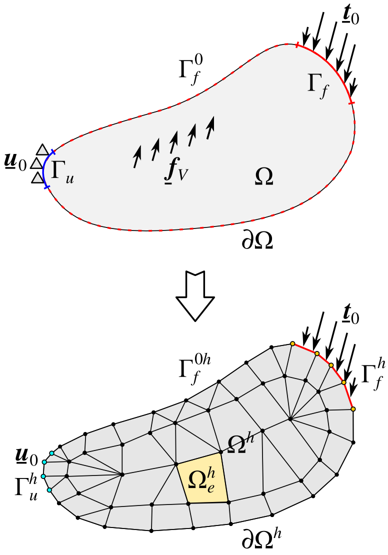
- Find displacement only on few locations $\underline{\boldsymbol{u}}_i(t)$ and interpolate in between

$$\underline{\boldsymbol{u}}(\underline{\boldsymbol{X}},t) = \sum N_i(\underline{\boldsymbol{X}})\underline{\boldsymbol{u}}_i(t)$$

- Thus, we reduce the problem of dimension ∞ to a finite dimensional problem
- Weak formulation of equilibrium equations results in a linear system of equations…
- Alternatively, the problem could be formulated as an optimization problem:

  Minimize body's potential energy for given external and internal loads

$$\min(U^h(\underline{\boldsymbol{u}}_i)) \text{ for } \underline{\boldsymbol{t}}_0 \text{ on } \Gamma_f^h \text{ and } \underline{\boldsymbol{u}}_0 \text{ on } \Gamma_u^h$$



Main idea

- From continuous to discrete problem
- Split solid into finite elements

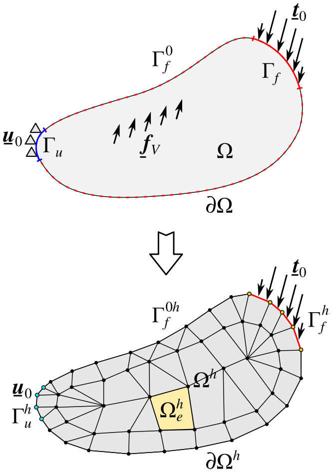
  $\Omega \rightarrow \Omega^h$ with $\Omega^h = \sum_e \Omega^h_e$

- All quantities are associated with this discretization:
  $\underline{u} \rightarrow \underline{u}^h$, $\underline{\underline{\sigma}} \rightarrow \underline{\underline{\sigma}}^h$, $\Gamma_f \rightarrow \Gamma^h_f$, $\underline{t}_0 \rightarrow \underline{t}^h_0$, $\ldots$
- Search for $\underline{u}^h$ only in a finite number of points (nodes)
- Interpolate in between (within elements)
- Ensure (1) equilibrium of every element and (2) satisfaction of boundary conditions

  (1) $\nabla \cdot \underline{\underline{\sigma}}^h + \underline{f}^h_v \sim 0$ in $\Omega^h_e, \forall e$

  (2.a) $\underline{\underline{\sigma}}^h \cdot \underline{n}^h \sim \underline{t}^h_0$ at Γ^h_f

  (2.b) $\underline{u}^h \sim \underline{u}^h_0$ at Γ^h_u



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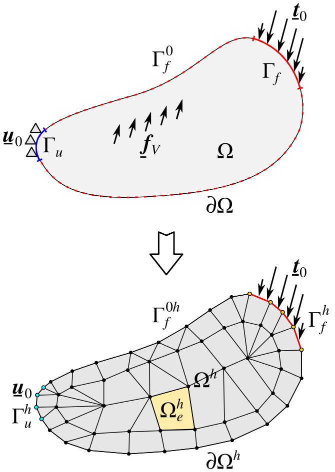

- Existence and uniqueness of the solution $\underline{u}^h_*$
- When discretization-size tends to zero $h \rightarrow 0$, convergence to the solution of the continuum problem: $\underline{u}^h_* \xrightarrow[h\rightarrow 0]{} \underline{u}_*$

Standard discrete system

1. For any discrete system the quantities of interest $[q]$ depend on system parameters $[p]$ and on locally acting external parameters $[e]$

$$[q]_i = [q]_i \left([p]_j, [e]_i\right)$$

2. In the first approximation this dependence is linear

$$\begin{aligned}
q_1 &= K_{11}p_1 + K_{12}p_2 + \dots K_{1N}p_N + A_{11}e_1 \\
q_2 &= K_{21}p_1 + K_{22}p_2 + \dots K_{2N}p_N + A_{22}e_2 \\
&\dots \\
q_N &= K_{21}p_1 + K_{22}p_2 + \dots K_{2N}p_N + A_{NN}e_N
\end{aligned}$$

3. In matrix form

$$[q]_i = [\mathbf{K}]_{ij}\,[p]_j + [\mathbf{A}]_{ii}\,[e]_i$$

4. Assuming that external parameters are of the same nature as quantities of interest ($[\mathbf{A}]_{ij} = [\mathbf{I}]_{ij}$)

$$\boxed{[q]_i = [\mathbf{K}]_{ij}\,[p]_j + [e]_i}$$

Discrete system in structural mechanics

Main quantities

- Quantities of interest $[q]$ are, in general, forces $[f]$
- System parameters $[p]$ are, in general, displacements $[u]$
- External parameters $[e]$ are, in general, external forces $[f]^{ext}$

Main steps

1. Construct *stiffness matrix* and *nodal loads* vector

$$[\mathbf{K}]_{ij}^{k}, [f]_{i}^{k}, \quad i,j \in 1, NN^{k}; k \in NE,$$

where NN^{k} is the number of nodes of k-th element, NE is the number of elements.

2. Assemble them into the global stiffness matrix and global load vector

$$[\mathbf{K}]_{ij}, [f]_{i}, \quad i,j \in 1, NN,$$

where NN is the total number of nodes.

3. Add boundary conditions (for example Dirichlet and Neumann)

$$[f]_{k}^{ext}, \quad k \in BC_f; \qquad [\mathbf{u}]_{l}^{0}, \quad l \in BC_u$$

4. Solve linear system of equations

$$[\mathbf{K}]_{ij}\,[\mathbf{u}]_{j} = [f]_{i} - [f]_{i}^{ext} \quad \rightarrow \quad [\mathbf{u}]_{j*}$$

Shape functions

- Displacements are known at nodes: $\underline{\boldsymbol{u}}_i^h,\ i = 1, 4$
- We need to know them inside the element
- Parametrize the inside with parameters $\{\xi, \eta\} \in [-1, 1]$
- Use *interpolation* or *shape* functions $N_i(\xi, \eta)$ for position $\underline{\boldsymbol{X}}$

$$\underline{\boldsymbol{X}}^h(\xi, \eta) = \sum_i \underline{\boldsymbol{X}}_i^h N_i(\xi, \eta)$$

and displacement $\underline{\boldsymbol{u}}$:

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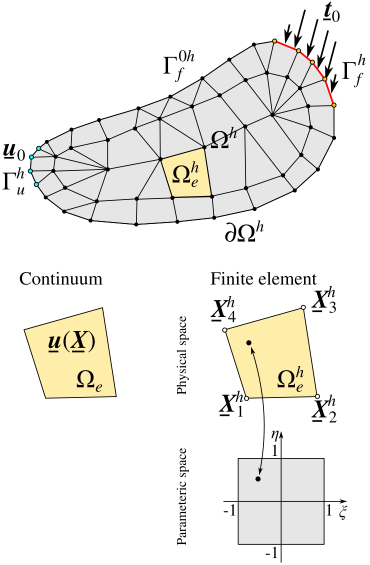
- If the same functions are used, then the element is called *isoparametric*
- Remark: Find $\{\xi, \eta\}$ from $\underline{\boldsymbol{X}}$ is not always straigthforward (may result in a system of non-linear equations)



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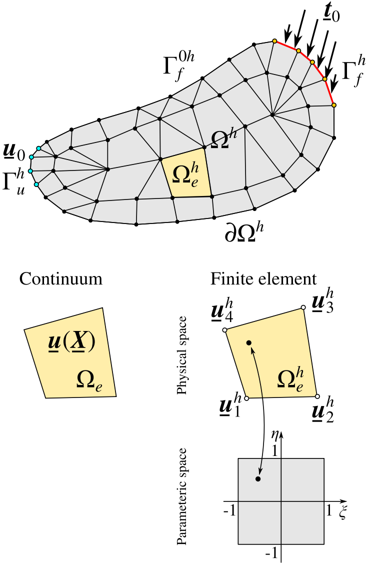


Shape functions II

Rules

- Node i has coordinates $\{\xi_i, \eta_i\}$
- Then $N_i(\xi_j, \eta_j) = \delta_{ij}$
- Partition of unity:

  $$\forall \xi, \eta, : \sum_i N_i(\xi, \eta) = 1$$

Types

- Linear shape functions

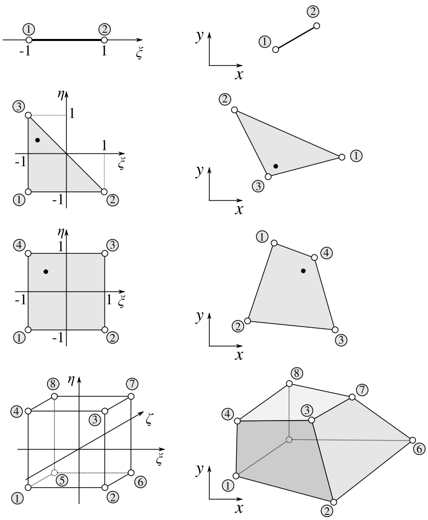
  $$\frac{\partial N}{\partial \xi} = \text{const}$$

- Non-linear shape functions

  $$\frac{\partial N}{\partial \xi} = f(\xi)$$

- Linear elements vs quadratic elements
- Higher order elements

Parametric space Physical space



Shape functions III

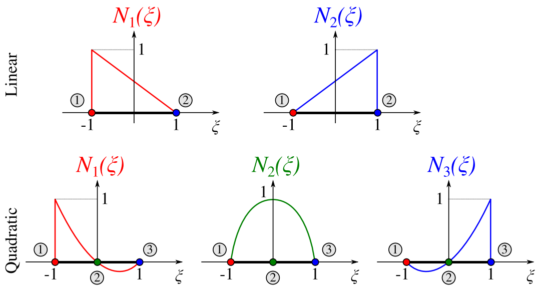
Example: bar element

- Linear shape functions:

$$N_1(\xi) = \frac{1}{2}(1 - \xi)$$
$$N_2(\xi) = \frac{1}{2}(1 + \xi)$$

- Quadratic shape functions:

$$N_1(\xi) = \frac{1}{2}\xi(\xi - 1)$$
$$N_2(\xi) = (1 - \xi^2)$$
$$N_3(\xi) = \frac{1}{2}\xi(1 + \xi)$$



Shape functions: vectors and matrices

- Displacement nodal vectors $\underline{u}_i = \underline{e}_x u_i^x + \underline{e}_y u_i^y$
- Array of nodal coordinates (size $\mathrm{dim} \cdot n$)

$$[\boldsymbol{X}] = [x_1,\ y_1,\ x_2,\ y_2,\ \dots\ x_n,\ y_n]^\intercal_{2n}$$

- Array of nodal displacements (size $\mathrm{dim} \cdot n$)

$$[\boldsymbol{u}] = [u_1^x,\ u_1^y,\ u_2^x,\ u_2^y,\ \dots\ u_n^x,\ u_n^y]^\intercal_{2n}$$

- Arrays of shape functions (size $\mathrm{dim} \cdot n$)

$$[\boldsymbol{N_x}] = [N_1,\ 0,\ N_2,\ 0,\ \dots\ N_n,\ 0]^\intercal_{2n}$$

$$[\boldsymbol{N_y}] = [0,\ N_1,\ 0,\ N_2,\ \dots\ 0,\ N_n]^\intercal_{2n}$$

$$[\boldsymbol{N}] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix}^\intercal_{2n\times\mathrm{dim}}$$

- Then

$$x(\xi,\eta,t) = [\boldsymbol{N_x(\xi,\eta)}]^\intercal [\boldsymbol{X(t)}], \quad y(\xi,\eta,t) = [\boldsymbol{N_y(\xi,\eta)}]^\intercal [\boldsymbol{X(t)}]$$

$$u^x(\xi,\eta,t) = [\boldsymbol{N_x(\xi,\eta)}]^\intercal [\boldsymbol{u(t)}], \quad u^y(\xi,\eta,t) = [\boldsymbol{N_y(\xi,\eta)}]^\intercal [\boldsymbol{u(t)}]$$

Gradients and shape functions

- Need to evaluate gradients (spatial derivatives) like $\frac{\partial f}{\partial x}$
- But with shape functions $f = f(\xi, \eta)$
- Then $\dfrac{\partial f(\xi,\eta)}{\partial x} = \dfrac{\partial f}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial \eta}{\partial x}$
- However, in general we do not have $\xi = \xi(x, y)$ but rather $x = x(\xi, \eta)$
- Let's do it other way around

$$\begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = [\,\mathbf{J}\,] \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix}$$

- Matrix $[\,\mathbf{J}\,]$ is called Jacobian operator/matrix and enables to obtain

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix}$$

Jacobian operator/matrix

- Jacobian operator/matrix:

$$[\mathbf{J}] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

- Using $x = [N_x]^\top [X], \quad y = [N_y]^\top [X]$ we get:

$$[\mathbf{J}] = \begin{bmatrix} [N_{x,\xi}]^\top [X] & [N_{y,\xi}]^\top [X] \\ [N_{x,\eta}]^\top [X] & [N_{y,\eta}]^\top [X] \end{bmatrix},$$

where $[N_{x,\xi}] = \left[\frac{\partial N_1}{\partial \xi}, \ 0, \ \frac{\partial N_2}{\partial \xi}, 0, \ \ldots \frac{\partial N_n}{\partial \xi}, 0 \right]^\top$ etc.

- Then the inverse Jacobian is given by:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta} \begin{bmatrix} [N_{y,\eta}]^\top [X] & -[N_{y,\xi}]^\top [X] \\ -[N_{x,\eta}]^\top [X] & [N_{x,\xi}]^\top [X] \end{bmatrix},$$

with the determinant of the Jacobian matrix (or simply Jacobian):

$$\Delta = \det([\mathbf{J}]) = [X]^\top \left([N_{x,\xi}][N_{y,\eta}]^\top - [N_{y,\xi}][N_{x,\eta}]^\top \right) [X] \neq 0$$

Infinitesimal strain in 2D

- Strain tensor: $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla\underline{u} + (\nabla\underline{u})^{\top}\right) \qquad (*)$

Infinitesimal strain in 2D

- Strain tensor: $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^{\intercal}\right) \qquad (*)$
- Interpolated displacements: $u^x = [\boldsymbol{N}_x]^{\intercal} [\boldsymbol{u}], \quad u^y = [\boldsymbol{N}_y]^{\intercal} [\boldsymbol{u}]$

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- Interpolated displacements: $u^x = [\boldsymbol{N}_x]^{\mathsf{T}}[\boldsymbol{u}], \quad u^y = [\boldsymbol{N}_y]^{\mathsf{T}}[\boldsymbol{u}]$
- Displacement gradient:

$$\nabla\underline{\boldsymbol{u}} = \frac{\partial\underline{\boldsymbol{u}}^h}{\partial x} \otimes \underline{\boldsymbol{e}}_x + \frac{\partial\underline{\boldsymbol{u}}^h}{\partial y} \otimes \underline{\boldsymbol{e}}_y$$

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$$(\nabla\underline{u})^{\mathsf{T}} \sim \begin{bmatrix} \frac{\partial u^x}{\partial x} & \frac{\partial u^y}{\partial x} \\ \frac{\partial u^x}{\partial y} & \frac{\partial u^y}{\partial y} \end{bmatrix}$$

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- Interpolated displacements: $u^x = [N_x]^\intercal [u], \quad u^y = [N_y]^\intercal [u]$
- Displacement gradient:

$$\nabla\underline{u} = \frac{\partial \underline{u}^h}{\partial x} \otimes \underline{e}_x + \frac{\partial \underline{u}^h}{\partial y} \otimes \underline{e}_y = \frac{\partial u^x}{\partial x}\underline{e}^x \otimes \underline{e}^x + \frac{\partial u^x}{\partial y}\underline{e}^x \otimes \underline{e}^y + \frac{\partial u^y}{\partial x}\underline{e}^y \otimes \underline{e}^x + \frac{\partial u^y}{\partial y}\underline{e}^y \otimes \underline{e}^y$$

$$(\nabla\underline{u})^\intercal \sim \begin{bmatrix} \frac{\partial u^x}{\partial x} & \frac{\partial u^y}{\partial x} \\ \frac{\partial u^x}{\partial y} & \frac{\partial u^y}{\partial y} \end{bmatrix} = [\,\mathrm{J}\,]^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} u^x \\ u^y \end{bmatrix}^\intercal$$

Infinitesimal strain in 2D

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$$(\nabla \underline{\boldsymbol{u}})^{\mathsf{T}} \sim \begin{bmatrix} \dfrac{\partial u^x}{\partial x} & \dfrac{\partial u^y}{\partial x} \\ \dfrac{\partial u^x}{\partial y} & \dfrac{\partial u^y}{\partial y} \end{bmatrix} = [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} u^x \\ u^y \end{bmatrix}^{\mathsf{T}} = [\,\mathbf{J}\,]^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} [\boldsymbol{N}_x]^{\mathsf{T}}[\boldsymbol{u}] \\ [\boldsymbol{N}_y]^{\mathsf{T}}[\boldsymbol{u}] \end{bmatrix}^{\mathsf{T}}$$

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Infinitesimal strain in 2D

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- Displacement gradient:

$$\nabla\underline{u} = \frac{\partial \underline{u}^h}{\partial x} \otimes \underline{e}_x + \frac{\partial \underline{u}^h}{\partial y} \otimes \underline{e}_y = \frac{\partial u^x}{\partial x}\underline{e}^x \otimes \underline{e}^x + \frac{\partial u^x}{\partial y}\underline{e}^x \otimes \underline{e}^y + \frac{\partial u^y}{\partial x}\underline{e}^y \otimes \underline{e}^x + \frac{\partial u^y}{\partial y}\underline{e}^y \otimes \underline{e}^y$$

$$(\nabla\underline{u})^{\intercal} \sim \begin{bmatrix} \dfrac{\partial u^x}{\partial x} & \dfrac{\partial u^y}{\partial x} \\ \dfrac{\partial u^x}{\partial y} & \dfrac{\partial u^y}{\partial y} \end{bmatrix} = [\,\boldsymbol{J}\,]^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} u^x \\ u^y \end{bmatrix}^{\intercal} = [\,\boldsymbol{J}\,]^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} [\boldsymbol{N}_x]^{\intercal}[\boldsymbol{u}] \\ [\boldsymbol{N}_y]^{\intercal}[\boldsymbol{u}] \end{bmatrix}^{\intercal} = [\,\boldsymbol{J}\,]^{-1} \begin{bmatrix} [\boldsymbol{N}_{x,\xi}]^{\intercal}[\boldsymbol{u}] & [\boldsymbol{N}_{y,\xi}]^{\intercal}[\boldsymbol{u}] \\ [\boldsymbol{N}_{x,\eta}]^{\intercal}[\boldsymbol{u}] & [\boldsymbol{N}_{y,\eta}]^{\intercal}[\boldsymbol{u}] \end{bmatrix}$$

- Represent strain tensor as an array (Voigt notations):

$$\underline{\underline{\varepsilon}} \quad \Rightarrow \quad [\boldsymbol{E}] = \begin{bmatrix} \varepsilon_{xx}, & \varepsilon_{yy}, & \gamma_{xy} \end{bmatrix}^{\intercal}, \quad \gamma_{xy} = 2\varepsilon_{xy}$$

Infinitesimal strain in 2D

- Strain tensor: $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^{\mathsf{T}}\right) \qquad (*)$
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- Then

$$[E] = \left[\frac{\partial u^x}{\partial x}, \quad \frac{\partial u^y}{\partial y}, \quad \frac{\partial u^y}{\partial x} + \frac{\partial u^x}{\partial y}\right]^{\mathsf{T}}$$

Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

$$[\,\mathbf{J}\,]^{-1} = \frac{1}{\Delta}\begin{bmatrix} [N_{y,\eta}]^{\top}[X] & -[N_{y,\xi}]^{\top}[X] \\ -[N_{x,\eta}]^{\top}[X] & [N_{x,\xi}]^{\top}[X] \end{bmatrix}$$

Infinitesimal strain in 2D in matrix form

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- Then the strain components are

$$\varepsilon_{xx} = \left(J_{11}[N_{x,\xi}] + J_{12}[N_{x,\eta}]\right)^{\intercal}[\boldsymbol{u}] = \frac{1}{\Delta}\left([N_{y,\eta}]^{\intercal}[X][N_{x,\xi}] - [N_{y,\xi}]^{\intercal}[X][N_{x,\eta}]\right)^{\intercal}[\boldsymbol{u}]$$

Infinitesimal strain in 2D in matrix form

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Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

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$$\varepsilon_{yy} = \left(J_{21}[\boldsymbol{N}_{y,\xi}] + J_{22}[\boldsymbol{N}_{y,\eta}]\right)^{\intercal}[\boldsymbol{u}] = \frac{1}{\Delta}\left(-[\boldsymbol{N}_{x,\eta}]^{\intercal}[\boldsymbol{X}][\boldsymbol{N}_{y,\xi}] + [\boldsymbol{N}_{x,\xi}]^{\intercal}[\boldsymbol{X}][\boldsymbol{N}_{y,\eta}]\right)^{\intercal}[\boldsymbol{u}]$$

Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta}\begin{bmatrix} [\boldsymbol{N}_{y,\eta}]^{\intercal}[\boldsymbol{X}] & -[\boldsymbol{N}_{y,\xi}]^{\intercal}[\boldsymbol{X}] \\ -[\boldsymbol{N}_{x,\eta}]^{\intercal}[\boldsymbol{X}] & [\boldsymbol{N}_{x,\xi}]^{\intercal}[\boldsymbol{X}] \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

- Then the strain components are

$$\varepsilon_{xx} = \left(J_{11}[\boldsymbol{N}_{x,\xi}] + J_{12}[\boldsymbol{N}_{x,\eta}]\right)^{\intercal}[\boldsymbol{u}] = \frac{1}{\Delta}\left([\boldsymbol{N}_{y,\eta}]^{\intercal}[\boldsymbol{X}][\boldsymbol{N}_{x,\xi}] - [\boldsymbol{N}_{y,\xi}]^{\intercal}[\boldsymbol{X}][\boldsymbol{N}_{x,\eta}]\right)^{\intercal}[\boldsymbol{u}] = [\boldsymbol{B}_1]^{\intercal}[\boldsymbol{u}]$$

$$\varepsilon_{yy} = \left(J_{21}[\boldsymbol{N}_{y,\xi}] + J_{22}[\boldsymbol{N}_{y,\eta}]\right)^{\intercal}[\boldsymbol{u}] = \frac{1}{\Delta}\left(-[\boldsymbol{N}_{x,\eta}]^{\intercal}[\boldsymbol{X}][\boldsymbol{N}_{y,\xi}] + [\boldsymbol{N}_{x,\xi}]^{\intercal}[\boldsymbol{X}][\boldsymbol{N}_{y,\eta}]\right)^{\intercal}[\boldsymbol{u}] = [\boldsymbol{B}_2]^{\intercal}[\boldsymbol{u}]$$

Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

$$[\,\mathbf{J}\,]^{-1} = \frac{1}{\Delta}\begin{bmatrix} [\boldsymbol{N}_{y,\eta}]^{\top}[\boldsymbol{X}] & -[\boldsymbol{N}_{y,\xi}]^{\top}[\boldsymbol{X}] \\ -[\boldsymbol{N}_{x,\eta}]^{\top}[\boldsymbol{X}] & [\boldsymbol{N}_{x,\xi}]^{\top}[\boldsymbol{X}] \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

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$$\varepsilon_{yy} = \left(J_{21}[\boldsymbol{N}_{y,\xi}] + J_{22}[\boldsymbol{N}_{y,\eta}]\right)^{\top}[\boldsymbol{u}] = \frac{1}{\Delta}\left(-[\boldsymbol{N}_{x,\eta}]^{\top}[\boldsymbol{X}][\boldsymbol{N}_{y,\xi}] + [\boldsymbol{N}_{x,\xi}]^{\top}[\boldsymbol{X}][\boldsymbol{N}_{y,\eta}]\right)^{\top}[\boldsymbol{u}] = [\boldsymbol{B_2}]^{\top}[\boldsymbol{u}]$$

$$\gamma_{xy} = \left(\frac{\partial u^x}{\partial y} + \frac{\partial u^y}{\partial x}\right) = \left(J_{11}[\boldsymbol{N}_{y,\xi}] + J_{12}[\boldsymbol{N}_{y,\eta}] + J_{21}[\boldsymbol{N}_{x,\xi}] + J_{22}[\boldsymbol{N}_{x,\eta}]\right)^{\top}[\boldsymbol{u}]$$

Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta}\begin{bmatrix} [\boldsymbol{N}_{y,\eta}]^\top[\boldsymbol{X}] & -[\boldsymbol{N}_{y,\xi}]^\top[\boldsymbol{X}] \\ -[\boldsymbol{N}_{x,\eta}]^\top[\boldsymbol{X}] & [\boldsymbol{N}_{x,\xi}]^\top[\boldsymbol{X}] \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

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$$\varepsilon_{yy} = \left(J_{21}[\boldsymbol{N}_{y,\xi}] + J_{22}[\boldsymbol{N}_{y,\eta}]\right)^\top[\boldsymbol{u}] = \frac{1}{\Delta}\left(-[\boldsymbol{N}_{x,\eta}]^\top[\boldsymbol{X}][\boldsymbol{N}_{y,\xi}] + [\boldsymbol{N}_{x,\xi}]^\top[\boldsymbol{X}][\boldsymbol{N}_{y,\eta}]\right)^\top[\boldsymbol{u}] = [\boldsymbol{B}_2]^\top[\boldsymbol{u}]$$

$$\gamma_{xy} = \left(\frac{\partial u^x}{\partial y} + \frac{\partial u^y}{\partial x}\right) = \left(J_{11}[\boldsymbol{N}_{y,\xi}] + J_{12}[\boldsymbol{N}_{y,\eta}] + J_{21}[\boldsymbol{N}_{x,\xi}] + J_{22}[\boldsymbol{N}_{x,\eta}]\right)^\top[\boldsymbol{u}]$$

$$\gamma_{xy} = \frac{1}{\Delta}\left([\boldsymbol{N}_{y,\eta}]^\top[\boldsymbol{X}][\boldsymbol{N}_{y,\xi}] - [\boldsymbol{N}_{y,\xi}]^\top[\boldsymbol{X}][\boldsymbol{N}_{y,\eta}] - [\boldsymbol{N}_{x,\eta}]^\top[\boldsymbol{X}][\boldsymbol{N}_{x,\xi}] + [\boldsymbol{N}_{x,\xi}]^\top[\boldsymbol{X}][\boldsymbol{N}_{x,\eta}]\right)^\top[\boldsymbol{u}]$$

Infinitesimal strain in 2D in matrix form

- ...continue. Jacobian matrix:

$$[\mathbf{J}]^{-1} = \frac{1}{\Delta}\begin{bmatrix} [N_{y,\eta}]^\top [X] & -[N_{y,\xi}]^\top [X] \\ -[N_{x,\eta}]^\top [X] & [N_{x,\xi}]^\top [X] \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

- Then the strain components are

$$\varepsilon_{xx} = \left(J_{11}[N_{x,\xi}] + J_{12}[N_{x,\eta}]\right)^\top [\boldsymbol{u}] = \frac{1}{\Delta}\left([N_{y,\eta}]^\top [X][N_{x,\xi}] - [N_{y,\xi}]^\top [X][N_{x,\eta}]\right)^\top [\boldsymbol{u}] = [\boldsymbol{B_1}]^\top [\boldsymbol{u}]$$

$$\varepsilon_{yy} = \left(J_{21}[N_{y,\xi}] + J_{22}[N_{y,\eta}]\right)^\top [\boldsymbol{u}] = \frac{1}{\Delta}\left(-[N_{x,\eta}]^\top [X][N_{y,\xi}] + [N_{x,\xi}]^\top [X][N_{y,\eta}]\right)^\top [\boldsymbol{u}] = [\boldsymbol{B_2}]^\top [\boldsymbol{u}]$$

$$\gamma_{xy} = \left(\frac{\partial u^x}{\partial y} + \frac{\partial u^y}{\partial x}\right) = \left(J_{11}[N_{y,\xi}] + J_{12}[N_{y,\eta}] + J_{21}[N_{x,\xi}] + J_{22}[N_{x,\eta}]\right)^\top [\boldsymbol{u}]$$

$$\gamma_{xy} = \frac{1}{\Delta}\left([N_{y,\eta}]^\top [X][N_{y,\xi}] - [N_{y,\xi}]^\top [X][N_{y,\eta}] - [N_{x,\eta}]^\top [X][N_{x,\xi}] + [N_{x,\xi}]^\top [X][N_{x,\eta}]\right)^\top [\boldsymbol{u}] = [\boldsymbol{B_3}]^\top [\boldsymbol{u}]$$

- Then

$$[E]_3 = [B]^\top_{3\times 2n} [\boldsymbol{u}]_{2n}$$

- With $[B] = \left[[\boldsymbol{B_1}]^\top, \quad [\boldsymbol{B_2}]^\top, \quad [\boldsymbol{B_3}]^\top\right]^\top$

Infinitesimal strain in 2D: example

- Consider a linear triangular element with shape functions:

  $$N_1 = -\tfrac{1}{2}(\xi + \eta), \quad N_2 = \tfrac{1}{2}(1 + \xi), \quad N_3 = \tfrac{1}{2}(1 + \eta)$$

- Their derivatives are given by:

  $$N_{1,\xi} = -1/2, \quad N_{2,\xi} = 1/2, \quad N_{3,\xi} = 0$$

  $$N_{1,\eta} = -1/2, \quad N_{2,\eta} = 0, \quad N_{3,\eta} = 1/2$$

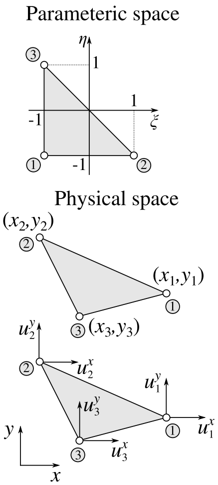
  $$\Delta = \frac{1}{4}((x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1))^{*}$$

- Then

  $$\varepsilon_{xx} = \frac{1}{4\Delta}\left[(y_3 - y_1)(u_2^x - u_1^x) - (y_2 - y_1)(u_3^x - u_1^x)\right]$$

  $$\varepsilon_{yy} = \frac{1}{4\Delta}\left[(x_2 - x_1)(u_3^y - u_1^y) - (x_3 - x_1)(u_2^y - u_1^y)\right]$$

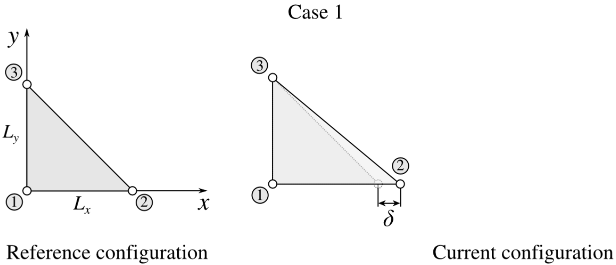
  $$\gamma_{xy} = \frac{1}{4\Delta}\left[(y_3 - y_1)(u_2^y - u_1^y) - (y_2 - y_1)(u_3^y - u_1^y) + (x_2 - x_1)(u_3^x - u_1^x) - (x_3 - x_1)(u_2^x - u_1^x)\right]$$



*Half of the area of the triangle.

Infinitesimal strain in 2D: example II

- Rectangular triangle $x_1 = x_3,\ y_1 = y_2,\ \Delta = L_x L_y/4$
- Case 1: pure tension/compression along OX iff $u_3^y = u_1^y,\ u_2^y = u_1^y,\ u_3^x = u_1^x$
  Ex.: $u_2^x = \delta :\quad \varepsilon_{xx} = \frac{1}{4\Delta}(y_3 - y_1)(u_2^x - u_1^x) = \delta/L_x, \quad \varepsilon_{yy} = \gamma_{xy} = 0$



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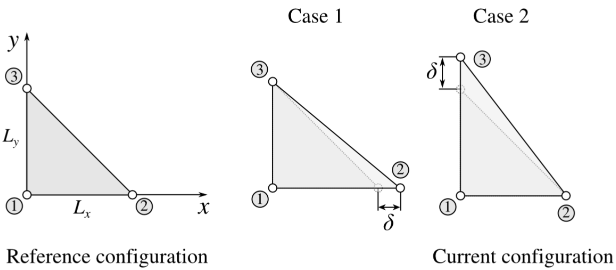
- Case 2: pure tension/compression along OY iff $u_2^x = u_1^x$, $u_2^y = u_1^y$, $u_3^x = u_1^x$
  Ex.: $u_3^y = \delta$: $\quad \varepsilon_{yy} = \frac{1}{4\Delta}(x_2 - x_1)(u_3^y - u_1^y) = \delta/L_y, \quad \varepsilon_{xx} = \gamma_{xy} = 0$



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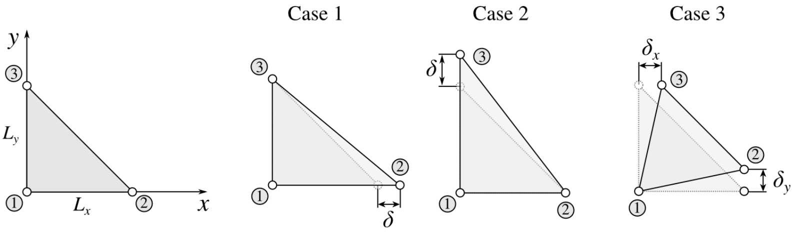
- Case 3: pure shear in XY iff $u_2^x = u_1^x,\ u_3^y = u_1^y$
  Ex.: $u_2^y = \delta_y,\ u_3^x = \delta_x:$

$$\gamma_{xy} = \frac{1}{4\Delta}\left((y_3 - y_1)(u_2^y - u_1^y) + (x_2 - x_1)(u_3^x - u_1^x)\right) = \frac{\delta_y}{L_x} + \frac{\delta_x}{L_y},\quad \varepsilon_{xx} = \varepsilon_{yy} = 0$$



Reference configuration

Current configuration

Stress tensor

- In linear elasticity, strain decomposition:

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}_{el} + \underline{\underline{\varepsilon}}_{th}$$

- With thermal strain field:

$$\underline{\underline{\varepsilon}}_{th} = \alpha(T - T_0)\underline{\underline{I}}$$

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$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}_{el} + \underline{\underline{\varepsilon}}_{th}$$

- With thermal strain field:

$$\underline{\underline{\varepsilon}}_{th} = \alpha(T - T_0)\underline{\underline{I}} = \alpha(\underline{X}) \left(T(\underline{X}) - T_0(\underline{X})\right)\underline{\underline{I}},$$

  where α is the coefficient of thermal expansion (CTE), T and T_0 are the current and reference temperature fields, respectively.

- The stress is defined by the elastic strain:

$$\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}_{th})$$

Stress: 2D isotropic elasticity

- Remind isotropic stress/strain relationship:

$$\underline{\underline{\sigma}} = \frac{\nu E}{(1+\nu)(1-2\nu)} \operatorname{tr}(\underline{\underline{\varepsilon}})\underline{\underline{\boldsymbol{I}}} + \frac{E}{1+\nu}\underline{\underline{\varepsilon}}$$

- Stress (in Voigt notations): $\underline{\underline{\sigma}} \quad \Rightarrow \quad [S] = [\sigma_{xx}, \quad \sigma_{yy}, \quad \sigma_{xy}]^{\top}$
- In plane stress $\sigma_{zz} = 0$, $\varepsilon_{zz} = \frac{\nu}{\nu-1}(\varepsilon_{xx} + \varepsilon_{yy})$
- In plain strain $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$, $\varepsilon_{zz} = 0$
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- Stress/strain relationship: $[S] = [\mathbf{D}]\,[E]$
- Matrix $[\mathbf{D}]$ in plane strain $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$:

$$[\mathbf{D}] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2^{*} \end{bmatrix}$$

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$$[\mathbf{D}] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2^{*} \end{bmatrix}$$

*Factor $1/2$ appears because $[E]$ contains γ_{xy} and not ε_{xy}.

Stress: general case

Voigt notations in 3D case

- Stress tensor: $\underline{\underline{\sigma}} \rightarrow [S] = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}]^\top$
- Strain tensor: $\underline{\underline{\varepsilon}} \rightarrow [E] = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}]^\top$
- Hooke's law: $[S] = [\mathbf{D}]\,[E]$

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- Hooke's law: $[S] = [\mathbf{D}]\,[E]$
- Isotropic elasticity (two constants E, ν):

$$[\mathbf{D}] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

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- Cubic elasticity (3 constants E, ν, μ):

$$[\mathbf{D}] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}$$

Stress: general case II

Voigt notations in 3D case

- Transversely isotropic elasticity (5 constants $E_1, E_2, \nu_1, \nu_2, \mu_1$):

$$[\mathbf{D}]_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 \end{bmatrix}$$

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- Orthotropic elasticity (9 constants $E_{xx}, E_{yy}, E_{zz}, \nu_{xy}, \nu_{yz}, \nu_{xz}, \mu_{xy}, \mu_{yz}, \mu_{xz}$):

$$[\mathbf{D}]_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

Equilibrium: from strong to weak form

- Equilibrium in absence of inertial forces

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f}_V = 0 \quad (*)$$

- Consistutive relation:

$$\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : \underline{\underline{\varepsilon}}$$

- Strain tensor:

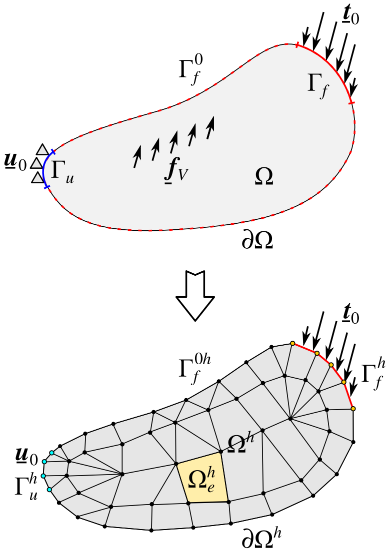
$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^{\top}\right)$$

- Boundary conditions (BC):

$\underline{u} = \underline{u}_0$ at Γ_u

$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_0$ at Γ_f

$\underline{\underline{\sigma}} \cdot \underline{n} = 0$ at Γ_f^0



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- Product with a virtual vector field $\underline{v}$ and integrate over a volume:

$$\int_{\Omega} (\nabla \cdot \underline{\underline{\sigma}}) \cdot \underline{v}\, dV + \int_{\Omega} \underline{f}_V \cdot \underline{v}\, dV = 0$$

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$$\boxed{\int_{\partial\Omega} \underline{\boldsymbol{n}} \cdot \underline{\underline{\sigma}} \cdot \underline{v}\, dS - \int_\Omega \underline{\underline{\sigma}} : (\nabla \underline{v})\, dV + \int_\Omega \underline{f}_V \cdot \underline{v}\, dV = 0}$$

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- If we select virtual vector field $\underline{v} = \delta \underline{u}$ as virtual displacements vanishing at Γ_u:

$$\boxed{\int_{\Gamma_f} \underline{t}_0 \cdot \delta \underline{u} \, dS - \int_{\Omega} \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}} \, dV + \int_{\Omega} \underline{f}_V \cdot \delta \underline{u} \, dV = 0}$$

- This variational formulation is called the *principle of virtual work* or of virtual displacements.

Stress and reactions: element's equilibrium II

- Work of imposed surface tractions on *virtual* displacements $= \frac{1}{2}\, \underline{t}_0 \cdot \delta \underline{u}$
- Work density of distributed volumetric forces $= \frac{1}{2}\, \underline{f}_V \cdot \delta \underline{u}$
- Corresponding virtual density of elastic energy $= \frac{1}{2}\, \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}}$

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- Corresponding virtual density of elastic energy $= \frac{1}{2}\, \underline{\underline{\boldsymbol{\sigma}}} : \delta \underline{\underline{\boldsymbol{\varepsilon}}}$
- According to the principle of virtual work:

$$\int_{\Omega} \underline{\underline{\boldsymbol{\sigma}}} : \delta \underline{\underline{\boldsymbol{\varepsilon}}}\, dV = \int_{\Gamma_f} \underline{\boldsymbol{t}}_0 \cdot \delta \underline{\boldsymbol{u}}\, dS + \int_{\Omega} \underline{\boldsymbol{f}}_V \cdot \delta \underline{\boldsymbol{u}}\, dV$$

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- Equivalently

$$\boxed{a(\underline{u}, \delta\underline{u}) = L(\delta\underline{u})}$$

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- Equivalently

$$\boxed{a(\underline{u}, \delta\underline{u}) = L(\delta\underline{u})}$$

with *bilinear form* $a(\underline{u}, \delta\underline{u}) = \int_\Omega \underline{\underline{\sigma}} : \nabla\delta\underline{u}\, dV = \int_\Omega \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}\, dV$

and *linear form* $L(\delta\underline{u}) = \int_{\Gamma_f} \underline{t}_0 \cdot \delta\underline{u}\, dS + \int_\Omega \underline{f}_V \cdot \delta\underline{u}\, dV$.

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- According to the principle of virtual work:

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- Equivalently

$$\boxed{a(\underline{u}, \delta \underline{u}) = L(\delta \underline{u})}$$

with *bilinear form* $a(\underline{u}, \delta \underline{u}) = \int_\Omega \underline{\underline{\sigma}} : \nabla \delta \underline{u}\, dV = \int_\Omega \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}}\, dV$

and *linear form* $L(\delta \underline{u}) = \int_{\Gamma_f} \underline{t}_0 \cdot \delta \underline{u}\, dS + \int_\Omega \underline{f}_V \cdot \delta \underline{u}\, dV$.

The functional space of kinematically admissible displacements and inducing finite energy is
$\underline{u} \in \mathbb{U} = \left\{ \underline{v} \in \mathbb{H}^1(\Omega) \,\middle|\, \underline{v} = \underline{u}_0 \text{ on } \Gamma_u \right\}$
whereas virtual displacements also inducing finite energy and vanishing at Dirichlet boundary belong to
$\delta \underline{u} \in \mathbb{V} = \left\{ \underline{v} \in \mathbb{H}^1(\Omega) \,\middle|\, \underline{v} = 0 \text{ on } \Gamma_u \right\}$
and $a : \mathbb{U} \times \mathbb{V} \to \mathbb{R}$ and $L : \mathbb{V} \to \mathbb{R}$, where $\mathbb{H}^1$ is the Hilbert space.

Stress and reactions: element's equilibrium II

- Work of imposed surface tractions on *virtual* displacements $= \frac{1}{2}\, \underline{t}_0 \cdot \delta \underline{u}$
- Work density of distributed volumetric forces $= \frac{1}{2}\, \underline{f}_V \cdot \delta \underline{u}$
- Corresponding virtual density of elastic energy $= \frac{1}{2}\, \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}}$
- According to the principle of virtual work:

$$\int_\Omega \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}}\, dV = \int_{\Gamma_f} \underline{t}_0 \cdot \delta \underline{u}\, dS + \int_\Omega \underline{f}_V \cdot \delta \underline{u}\, dV$$

- Equivalently

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- So we are in the framework of the *Lax-Milgram theorem* (continuity and coercivity could be easily shown).

Stress and reactions: element's equilibrium II

- According to the principle of virtual work:

$$\int_{\Omega} \underline{\underline{\boldsymbol{\sigma}}} : \delta \underline{\underline{\boldsymbol{\varepsilon}}} \, dV - \int_{\Omega} \underline{\boldsymbol{f}}_V \cdot \delta \underline{\boldsymbol{u}} \, dV = \int_{\Gamma_f} \underline{\boldsymbol{t}}_0 \cdot \delta \underline{\boldsymbol{u}} \, dS$$

Stress and reactions: element's equilibrium II

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- Elastic stress $\underline{\underline{\sigma}} = {}^4\underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}_{th}) \Rightarrow [S] = [\mathbf{D}]\,([E] - [E_{th}])$
- Strain $\underline{\underline{\varepsilon}} \sim [E] = [B]^{\intercal}\,[u]$
- Volumetric force density $\underline{f}_v \sim [f_v] = [f_v^x, f_v^y, f_v^z]^{\intercal}$
- Virtual displacement $\delta \underline{u} \sim [N]^{\intercal}\,\delta[u]$

Stress and reactions: element's equilibrium II

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- Virtual displacement $\delta\underline{\boldsymbol{u}} \sim [N]^{\mathsf{T}}\,\delta[\boldsymbol{u}]$
- The discretized form of the virtual work:

$$\int_{\Omega^h} \left\{ ([\mathbf{D}]\,([E] - [E_{th}]))^{\mathsf{T}}\, \delta[E] - [f_v]^{\mathsf{T}}\,[N_i]^{\mathsf{T}}\, \delta[\boldsymbol{u}] \right\} dV = \int_{\Gamma_f^h} \underline{t}_0(\underline{X})[N_i]^{\mathsf{T}}\, dS\, \delta[\boldsymbol{u}]$$

Stress and reactions: element's equilibrium II

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$$\int_{\Omega} \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}} \, dV - \int_{\Omega} \underline{f}_V \cdot \delta \underline{u} \, dV = \int_{\Gamma_f} \underline{t}_0 \cdot \delta \underline{u} \, dS$$

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- The discretized form of the virtual work:

$$\int_{\Omega^h} \left\{ ([\mathbf{D}]\,([E] - [E_{th}]))^\top \delta[E] - [f_v]^\top [N_i]^\top \delta[\boldsymbol{u}] \right\} dV = \int_{\Gamma_f^h} \underline{t}_0(\underline{X})[N_i]^\top \, dS \, \delta[\boldsymbol{u}]$$

$$[\boldsymbol{u}] \left[\int_{\Omega^h} [B]\,[\mathbf{D}]\,[B]^\top \, dV \right] \delta[\boldsymbol{u}] - \left[\int_{\Omega^h} \left([f_v]^\top [N_i]^\top + [E_{th}]^\top \, [\mathbf{D}]\,[B]^\top \right) dV \right] \delta[\boldsymbol{u}] = [f]^\top \delta[\boldsymbol{u}]$$

Stress and reactions: element's equilibrium II

- Balance of virtual work for a single element:

$$[\boldsymbol{u}]\left[\int_{\Omega^h}[B]\,[\mathbf{D}]\,[B]^{\mathsf{T}}\,dV\right]\delta[\boldsymbol{u}]-\left[\int_{\Omega^h}\left([f_v]^{\mathsf{T}}\,[N_i]^{\mathsf{T}}+[E_{th}]^{\mathsf{T}}\,[\mathbf{D}]\,[B]^{\mathsf{T}}\right)dV\right]\delta[\boldsymbol{u}]=[f]^{\mathsf{T}}\,\delta[\boldsymbol{u}]$$

- For arbitrary virtual displacements $\delta[\boldsymbol{u}]$:

$$\underbrace{\left[\int_{V^e}[B]^{\mathsf{T}}\,[\mathbf{D}]\,[B]\,dV\right]}_{[\mathbf{K}^{\mathbf{e}}]}[\boldsymbol{u}]+\underbrace{\left[\int_{V^e}\left(-[f_v]^{\mathsf{T}}\,[N_i]^{\mathsf{T}}-[B]\,[\mathbf{D}]\,[E_{th}]\right)dV\right]}_{[f^e_{int}]}=\underbrace{[f]}_{[f^e_{ext}]}$$

- System of equations linking displacements and reactions:

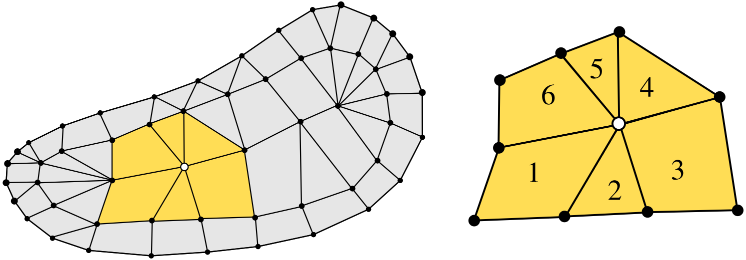
$$\boxed{[\mathbf{K}^{\mathbf{e}}]\,[u^e]+[f^e_{int}]=[f^e_{ext}]}$$

Assembly

- At every internal node the total force should be zero:

$$\sum_e [f^e_{ext}] = 0$$

summation over all elements e attached to this node.



- Summation over all nodes gives:

$$[\mathbf{K}]\,[u] + [f_{int}] = 0$$

Dirichlet boundary conditions

Dirichlet BC

- Use penalty method to enforce prescribed displacements: array $[\boldsymbol{u_0}] = [0\ldots0\ u_{i0}\ 0\ldots0\ u_{j0}\ 0]$
- Diagonal selection matrix $[\mathbf{I^s}]$ with ones at prescribed degrees of freedom (DOFs):

$$[\mathbf{I^s}] = \begin{array}{c} \begin{array}{ccccccccc} & & & \overbrace{}^{i} & & & & \overbrace{}^{j} & \end{array} \\ \begin{bmatrix} 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \} \; i \\ \\ \\ \\ \\ \} \; j \\ \\ \end{array}$$

- Then the system is changed to

$$([\mathbf{K}] + \epsilon\,[\mathbf{I^s}])\,[u] = [f_{ext}] - [f_{int}] + \epsilon[\boldsymbol{u_0}]$$

  where ϵ is the penalty coefficient such that $\epsilon \gg \max(K_{ij})$, and $[\mathbf{I}]$ is the identity matrix.
- Alternatively, (i) a direct DOF elimination or (ii) Lagrange multipliers could be used.

Neumann boundary conditions

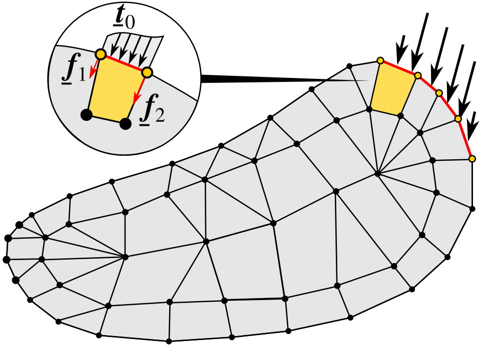
Neumann BC

- Surface traction $\underline{t}_0$ at Γ_f
- Virtual work of surface traction over one element:

$$\int_{\Gamma_f^e} \underline{t}_0 \cdot \delta \underline{u}\, d\Gamma = \underline{f}^{i}_{ext} \cdot \delta \underline{u}^{e}_{i}$$

- Then

$$[\boldsymbol{f}^{\boldsymbol{i}}_{ext}] = \int_{\Gamma_f^e} [\boldsymbol{t}_0][N]^{\top}\, d\Gamma$$



Discrete system of equations

- Balance of virtual work for the whole body:

$$\underbrace{\left[\int_V [B]^{\intercal}\,[\mathbf{D}]\,[B]\,dV\right]}_{[\mathbf{K}]}[u] = \underbrace{\int_{\Gamma_f} [t_0]^{\intercal}\,[N]^{\intercal}\,d\Gamma}_{[f_{ext}]} + \underbrace{\left[\int_V \left([f_v]^{\intercal}\,[N_i]^{\intercal} + [B]\,[\mathbf{D}]\,[E_{th}]\right) dV\right]}_{-[f_{int}]}$$

- System of equations linking displacements and reactions:

$$\boxed{[\mathbf{K}]\,[u] = [f_{ext}] - [f_{int}]}$$

- Stiffness matrix $[\mathbf{K}]$
- Vector of degrees of freedom (DOFs) $[u]$
- Right hand term (vector of forces) $[f_{ext}] - [f_{int}]$

Evaluation of the integrals

- Weak form (recall):

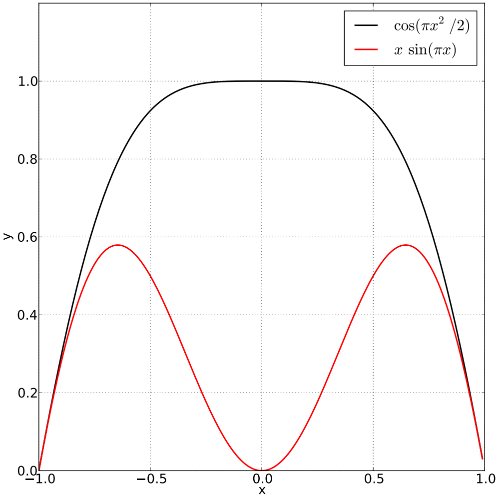
$$\underbrace{\left[\int_V [B]^\top [\mathbf{D}] [B]\, dV\right]}_{[\mathbf{K}]} [u] = \underbrace{\int_{\Gamma_f} [t_0]^\top [N]^\top\, d\Gamma}_{[f_{ext}]} + \underbrace{\left[\int_V \left([f_v]^\top [N_i]^\top + [B] [\mathbf{D}] [E_{th}]\right) dV\right]}_{-[f_{int}]}$$

- Exact integration: $\int_a^b f(x)dx = F(b) - F(a)$ (not always possible)
- Approximate integration (trapezoidal rule, Simpson's rule)
- Gauss quadrature: $\int_a^b f(x)dx \approx \sum_{i=1}^{N_{GP}} w_i f(x_i)$
- Gauss points x_i and weights w_i with $i = 1, N_{GP}$
- Integration is exact for polynomials of order $2N_{GP} - 1$
- Tabulated data for x_i, w_i (1D, 2D, 3D integration)

Evaluation of the integrals: example

- Function $f(x) = \cos(\pi x^2/2)$
  - $N_{GP} = 1$: error ≈ 28.22 %
  - $N_{GP} = 2$: error ≈ 11.04 %
  - $N_{GP} = 3$: error ≈ 1.14 %
  - $N_{GP} = 4$: error ≈ 0.14 %
  - $N_{GP} = 5$: error ≈ 0.01 %

- Function $f(x) = x\ \sin(\pi x)$
  - $N_{GP} = 1$: error ≈ 100.00 %
  - $N_{GP} = 2$: error ≈ 76.05 %
  - $N_{GP} = 3$: error ≈ 12.07 %
  - $N_{GP} = 4$: error ≈ 0.80 %
  - $N_{GP} = 5$: error ≈ 0.03 %



Evaluation of the integrals II

- Consider: $\int_V [\boldsymbol{B}]^\top [\mathbf{D}] [\boldsymbol{B}]\, dV = \sum_{e=1}^{N_e} \int_{V_e} [\boldsymbol{B}]^\top [\mathbf{D}] [\boldsymbol{B}]\, dV$
- Transpose to the parametric space or mapping (in 2D case):

$$\int_{V_e} [\boldsymbol{B}(\xi,\eta)]^\top [\mathbf{D}] [\boldsymbol{B}(\xi,\eta)]\, dV = \int_{-1}^{1}\int_{-1}^{1} [\boldsymbol{B}(\xi,\eta)]^\top [\mathbf{D}] [\boldsymbol{B}(\xi,\eta)] \det([\,\mathbf{J}\,])\, d\xi d\eta$$

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- Finally:

$$[\mathbf{K}] = \int_V [B]^\top [\mathbf{D}] [B] \, dV \approx \sum_{e=1}^{N_e} \sum_{GP=1}^{N_{GP}} [\boldsymbol{B}^e(\xi_{GP},\eta_{GP})]^\top [\mathbf{D}] [\boldsymbol{B}^e(\xi_{GP},\eta_{GP})] \det([\, J^e(\xi_{GP},\eta_{GP}) \,]) w_{GP}$$

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Evaluation of the integrals III

- If $N(\xi,\eta) = P_p$ is a polynomial of order p, then $[\,\mathbf{J}\,] = P_{(p-1)}$, $[\boldsymbol{B}] = \frac{P_{2(p-1)}}{Q_{(p-1)}}$
- **Remark I:** Gauss quadrature is exact for $p = 1$ and approximate if $p > 1$.
- **Remark II:** Stress and strains are exactly evaluated only in Gauss points, in all other points they are extrapolated/interpolated
- **Remark III:** Underintegration may lead to zero-energy deformation modes (which are often stabilized in FE software)

Evaluation of the integrals: quadrilateral 2D element

- Shape functions:

$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \tfrac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$

- Shape function derivatives:

$$N_{1,\xi} = -\tfrac{1}{4}(1-\eta), \quad N_{2,\xi} = \tfrac{1}{4}(1-\eta)$$
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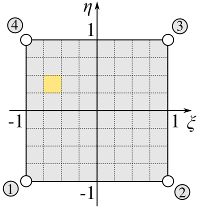
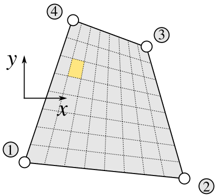
- Determinant of Jacobian ($dA = \det[\,\mathbf{J}\,]d\xi d\eta$):

$$\det([\,\mathbf{J}\,]) =$$
$$\tfrac{1}{16}\Big[\Big((1-\eta)(x_2-x_1)+(1+\eta)(x_3-x_4)\Big)\Big((1+\xi)(y_3-y_2)+(1-\xi)(y_4-y_1)\Big)-$$
$$-\;((1-\eta)(y_2-y_1)+(1+\eta)(y_3-y_4))((1+\xi)(x_3-x_2)+(1-\xi)(x_4-x_1))\Big]$$

Parametric space



Physical space



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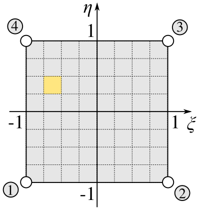
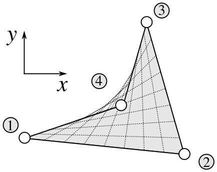
$$-\;((1-\eta)(y_2-y_1)+(1+\eta)(y_3-y_4))((1+\xi)(x_3-x_2)+(1-\xi)(x_4-x_1))\Big]$$

- **Warning:** to ensure $\det([\,\mathbf{J}\,]) > 0$ the element should remain convex

Parametric space



Physical space



Solvers

Problem: Find $[u]$ such that $\boxed{[\mathbf{K}]\,[u] = [f]}$, i.e. $[u] = [\mathbf{K}]^{-1}\,[f]$

- **Iterative solvers**
  The solution is approached iteratively, does not require much memory, restrictions to matrix type, sensitive to matrix conditioning, a preconditioner is often needed.
  - Gauss-Seidel method (GS)
  - Conjugate gradient method (CG)
  - Generalized minimum residual method (GMRES)
  - . . .
- **Direct solvers**
  The solution is provided directly, no restrictions on matrix type, less sensitive to matrix conditioning, based on LU *or Cholesky decomposition*
  - Frontal
  - Sparse direct
  - . . .

Convergence

Mesh and interpolation order convergence

- For Sobolev spaces[1] $\underline{\boldsymbol{u}} \in \mathbb{W}^{s,p}$, $s, p \in \mathbb{N}$ and their norm: $||\underline{\boldsymbol{u}}||_{\mathbb{W}^{s,p}} = \left[\int_\Omega \sum_{\alpha=0}^{s} \left(\frac{\partial^\alpha \underline{\boldsymbol{u}}}{\partial \underline{\boldsymbol{x}}^\alpha} \cdot \frac{\partial^\alpha \underline{\boldsymbol{u}}}{\partial \underline{\boldsymbol{x}}^\alpha} \right)^p dV \right]^{1/p}$
- For Hilbert space $\mathbb{H}^1$:

$$||\underline{\boldsymbol{u}}||_{\mathbb{H}^1} = \sqrt{\int_\Omega \left(\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{u}} + l^2 \nabla \underline{\boldsymbol{u}} : \nabla \underline{\boldsymbol{u}} \right) dV}$$

[1]The solution is usually sought in physically meaningful Sobolev space $W^{1,2}$, i.e. Hilbert space $\mathbb{H}^1$.

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- If there's no re-entrant corners and boundary conditions are "gentle", then displacements converge as :

$$\frac{\|\underline{\boldsymbol{u}} - \underline{\boldsymbol{u}}^h\|_{H^0}}{\|\underline{\boldsymbol{u}}\|_{H^0}} \leq C_u h^{p+1}$$

  where $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{u}}^h$ are the true and approximate solutions, p is the interpolation order of shape functions $N(\xi, \eta)$ and h is the element size.

[1] The solution is usually sought in physically meaningful Sobolev space $W^{1,2}$, i.e. Hilbert space $\mathbb{H}^1$.

Mesh and interpolation order convergence

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- If there's no re-entrant corners and boundary conditions are "gentle", then displacements converge as :

$$\frac{||\underline{u} - \underline{u}^h||_{H^0}}{||\underline{u}||_{H^0}} \leq C_u h^{p+1}$$

where $\underline{u}$, $\underline{u}^h$ are the true and approximate solutions, p is the interpolation order of shape functions $N(\xi, \eta)$ and h is the element size.

- And that stresses/strains converge as:

$$\frac{||\underline{u} - \underline{u}^h||_{H^1}}{||\underline{u}||_{H^1}} \leq C_\sigma h^p$$

[1] The solution is usually sought in physically meaningful Sobolev space $W^{1,2}$, i.e. Hilbert space $\mathbb{H}^1$.

Mesh and interpolation order convergence

- For Sobolev spaces[1] $\underline{u} \in \mathbb{W}^{s,p}$, $s, p \in \mathbb{N}$ and their norm: $\|\underline{u}\|_{\mathbb{W}^{s,p}} = \left[\int\limits_\Omega \sum_{\alpha=0}^{s} \left(\frac{\partial^\alpha \underline{u}}{\partial \underline{x}^\alpha} \cdot \frac{\partial^\alpha \underline{u}}{\partial \underline{x}^\alpha}\right)^p dV\right]^{1/p}$
- For Hilbert space $\mathbb{H}^1$:

$$\|\underline{u}\|_{\mathbb{H}^1} = \sqrt{\int\limits_\Omega \left(\underline{u} \cdot \underline{u} + l^2 \nabla \underline{u} : \nabla \underline{u}\right) dV} = \sqrt{\int\limits_\Omega \left(\underline{u} \cdot \underline{u} + \nabla \underline{u} : \nabla \underline{u}\right) dV}$$

- If there's no re-entrant corners and boundary conditions are "gentle", then displacements converge as :

$$\frac{\|\underline{u} - \underline{u}^h\|_{H^0}}{\|\underline{u}\|_{H^0}} \leq C_u h^{p+1}$$

  where $\underline{u}$, $\underline{u}^h$ are the true and approximate solutions, p is the interpolation order of shape functions $N(\xi, \eta)$ and h is the element size.
- And that stresses/strains converge as:

$$\frac{\|\underline{u} - \underline{u}^h\|_{H^1}}{\|\underline{u}\|_{H^1}} \leq C_\sigma h^p$$

- Therefore, to obtain a converged solution we can either increase interpolation order p (**p-refinement**) or decrease h (**h-refinement**)

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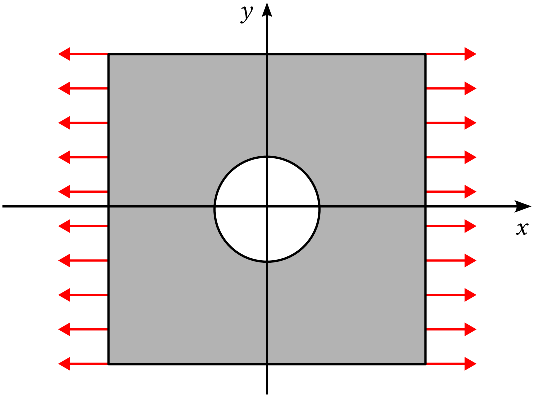
[1]The solution is usually sought in physically meaningful Sobolev space $W^{1,2}$, i.e. Hilbert space $\mathbb{H}^1$.

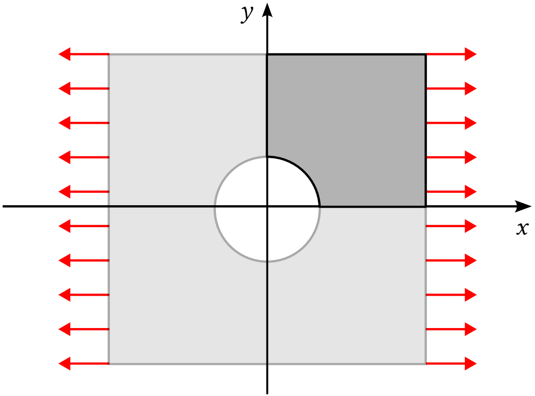
Example

Tension of a rectangular sheet with a hole



Example

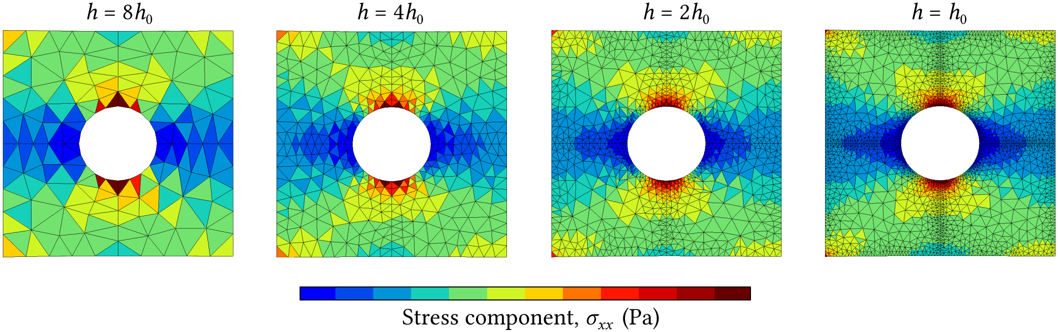
Tension of a rectangular sheet with a hole



The symmetry is used to reduce the computational cost☺

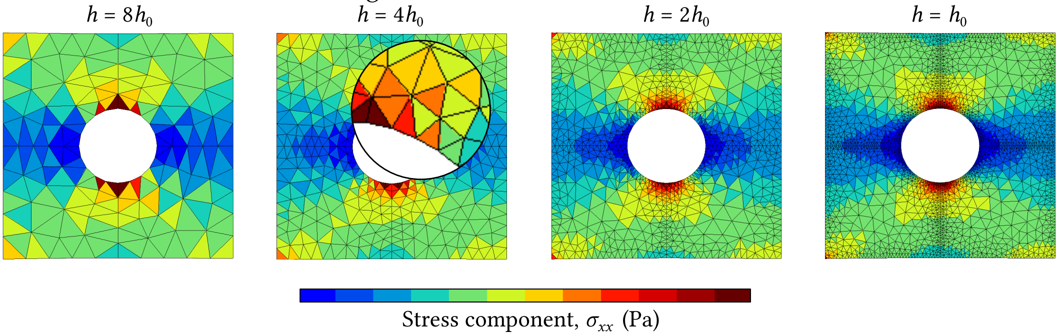
Example

Triangular mesh with linear elements :



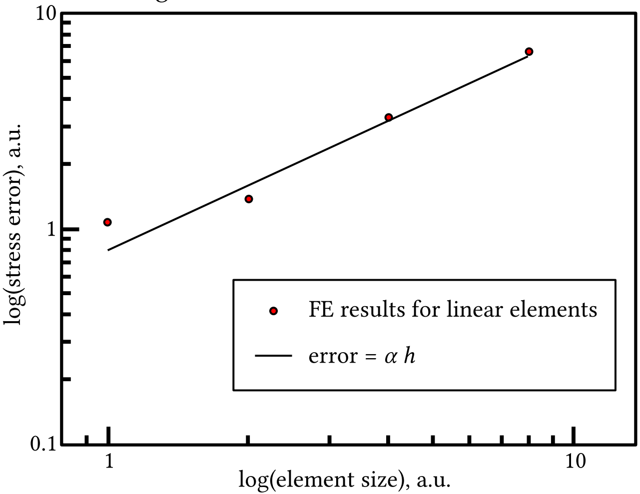
Example

Triangular mesh with linear elements :



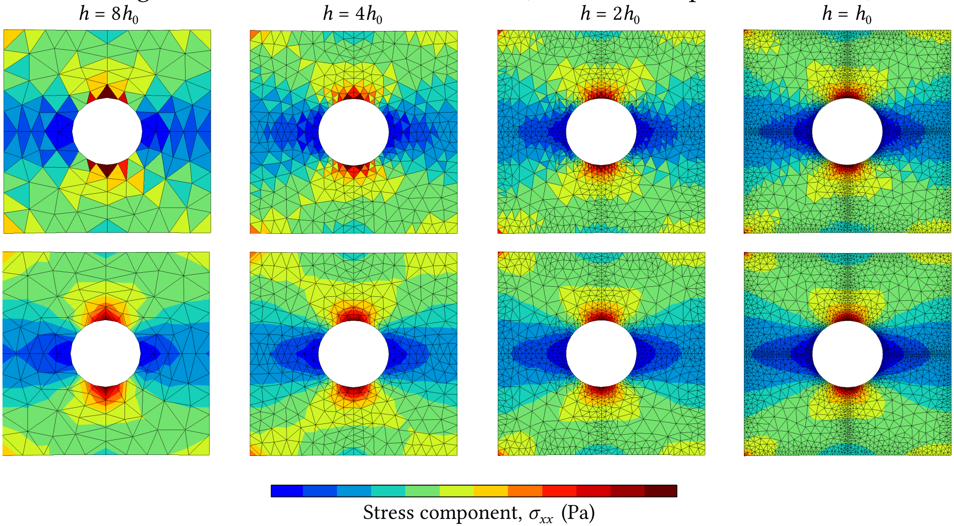
Example

Triangular mesh with linear elements :



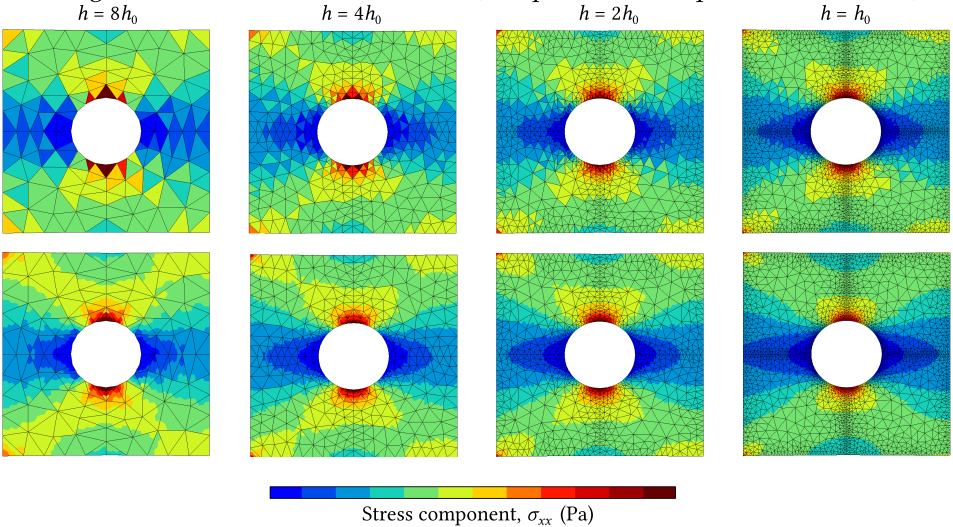
Example

Triangular mesh with linear elements (with contour plot stress field):



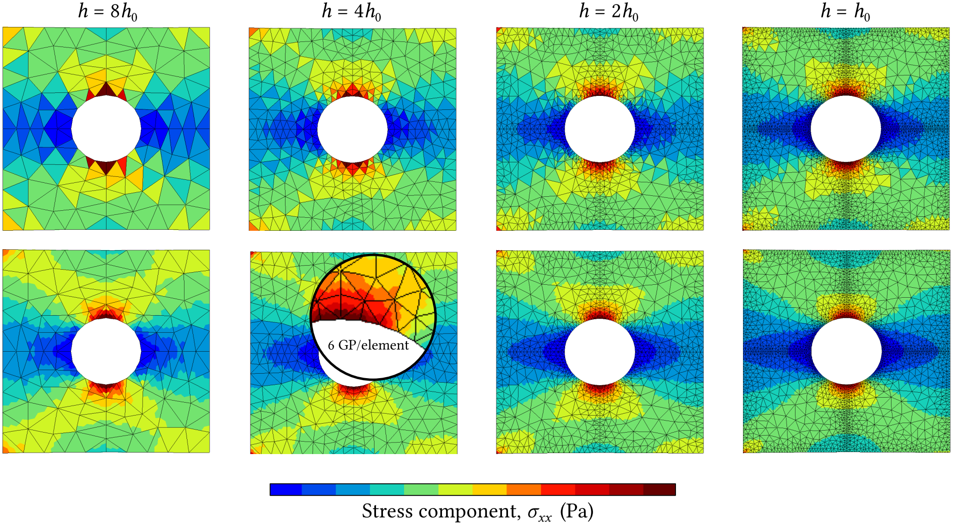
Example

Triangular mesh with linear elements (comparison with quadratic elements):



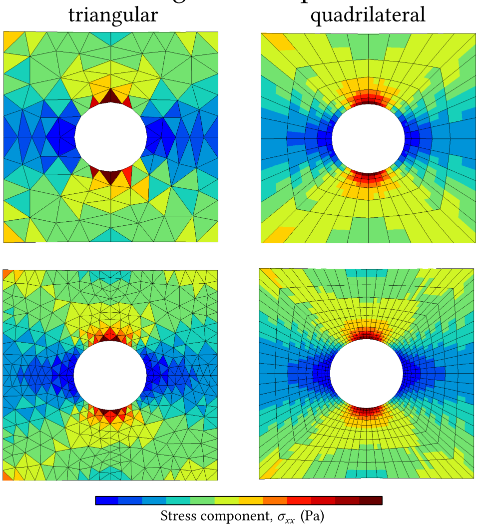
Example

Triangular mesh with linear elements (comparison with quadratic elements):



Example

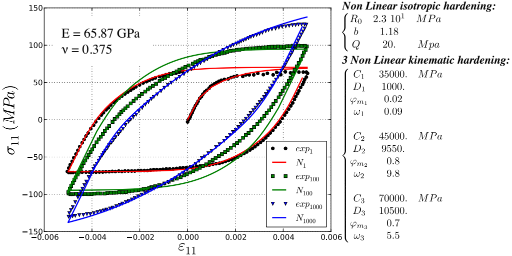
Comparison of triangular and quadrilateral meshes:

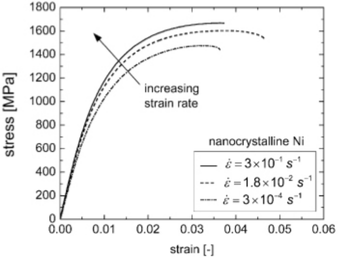


Nonlinear FEM

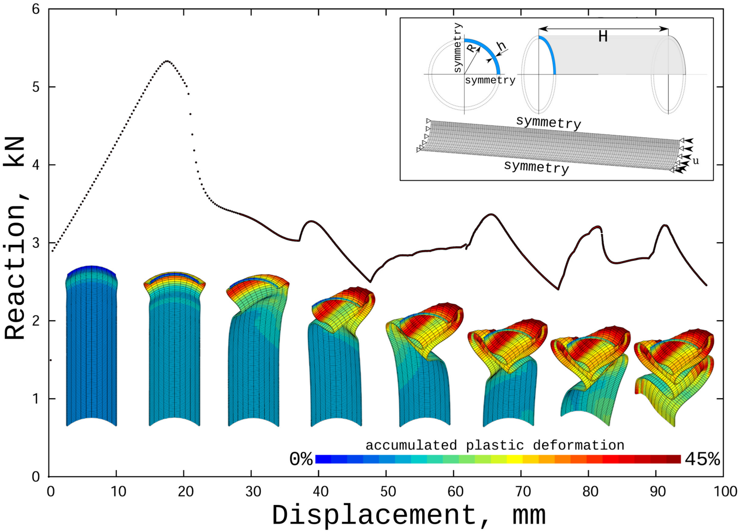
Types of nonlinearity

- Material behavior (viscoelasticity, plasticity, damage)
- Nonlinear geometry = finite deformations and/or rotations $\Omega(t) \neq \Omega(t_0)$, infinitesimal strain tensor $\underline{\underline{\varepsilon}}$ is not applicable
- Fracture (crack propagation: remeshing of X-FEM)
- Contact, friction, wear
- Coupled thermomecanical or fluid/solid problems

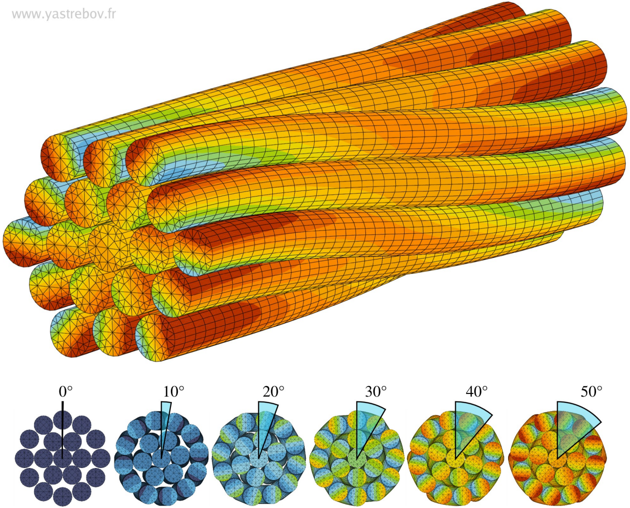




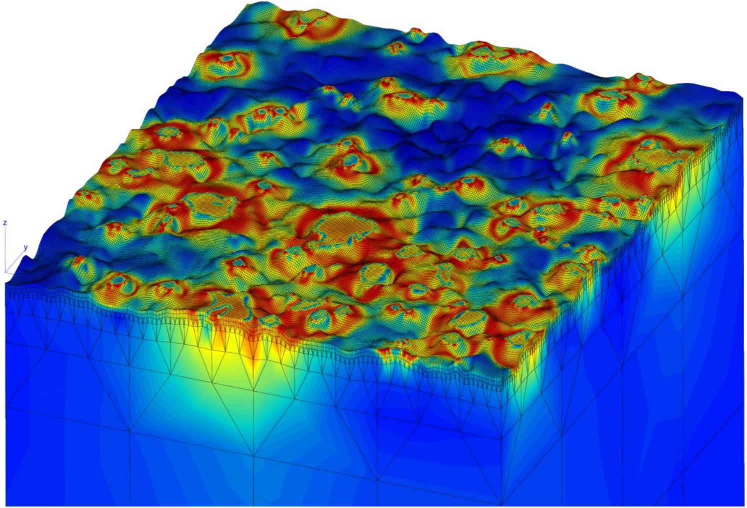
Post-buckling behavior with self-contact



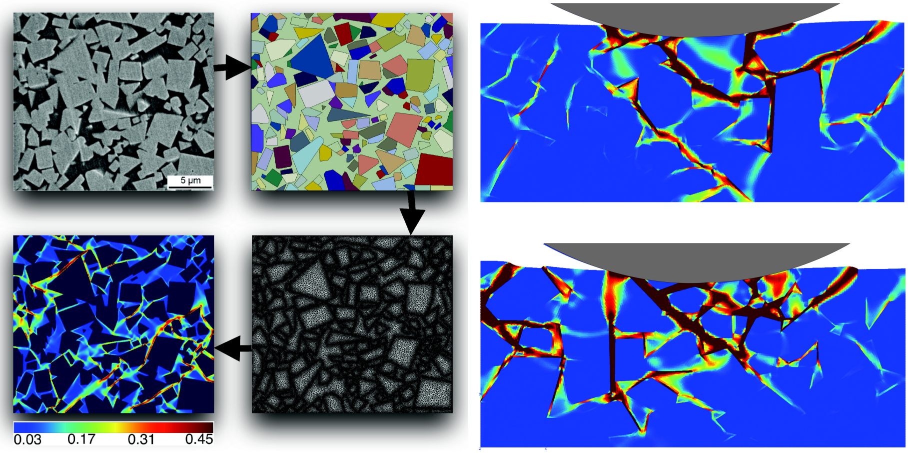
Twisting multi-strand wire



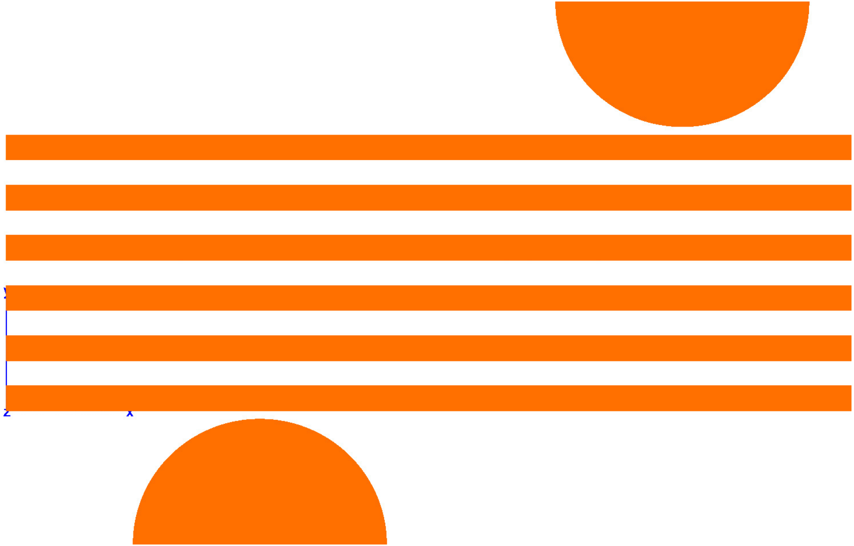
Contact of a rough surface



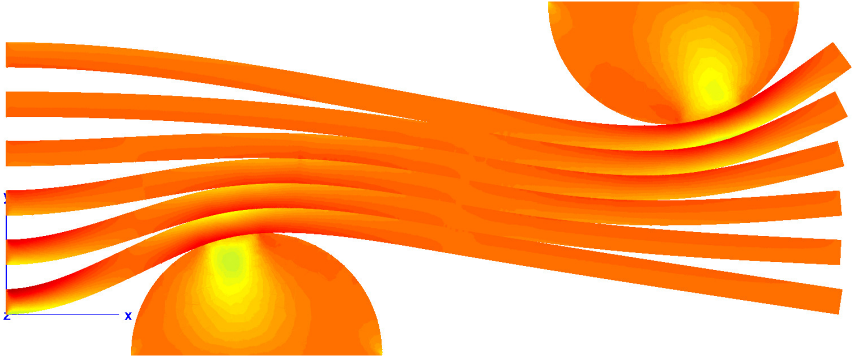
Impact of WC/Co composite



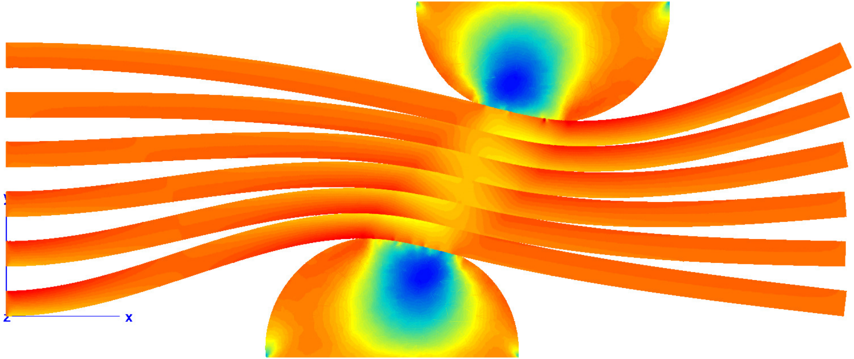
Multi-contact problem



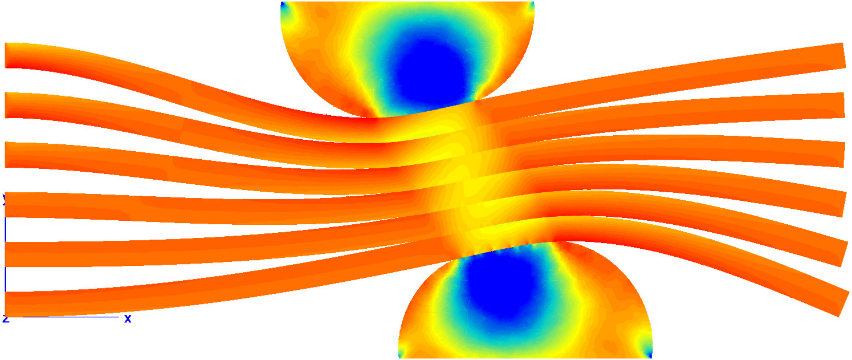
Multi-contact problem



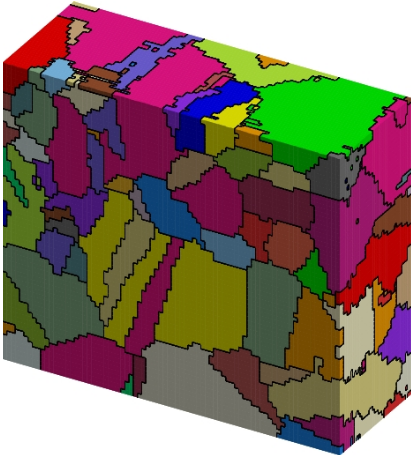
Multi-contact problem

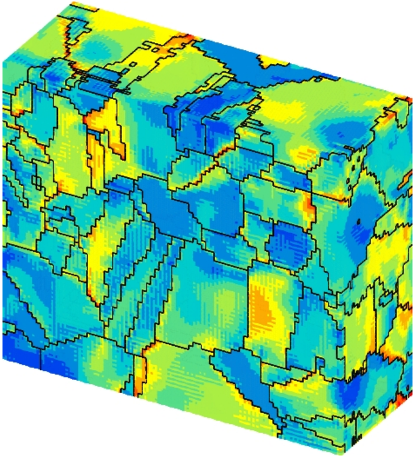


Multi-contact problem

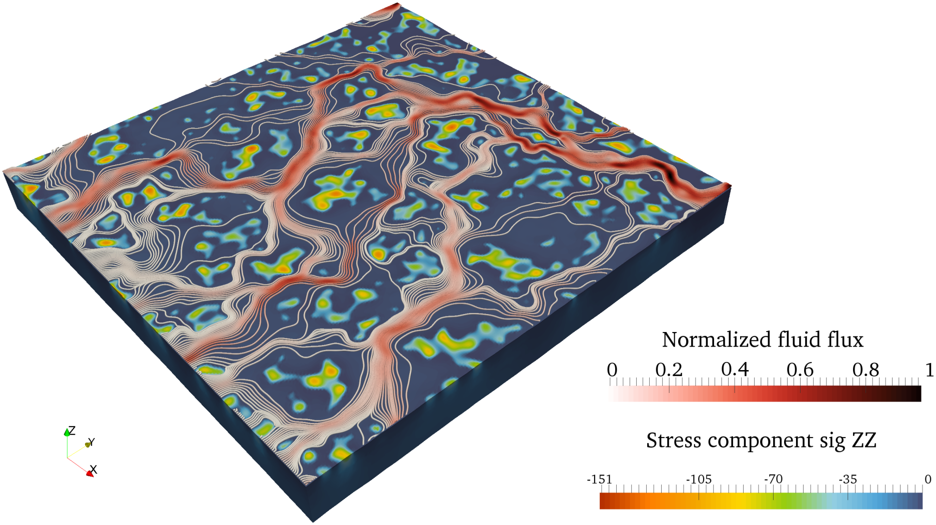


Polycristalline material





Coupled thin flow in contact interface



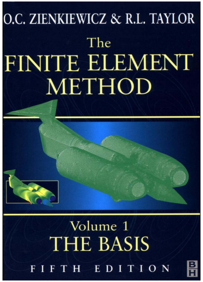
Conclusion

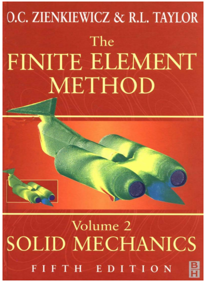
- The linear Finite Element Method is widely used in mechanical engineering
- To get to a matrix formulation (linear system of equations)

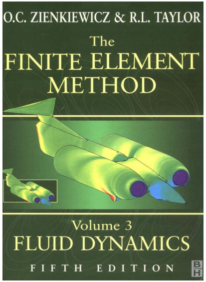
$$[K][u] = [f]$$

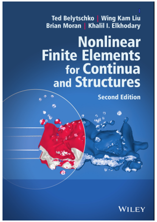
  we need to compute:
  - a matrix $[B]$ at every Gauss point (GP)
  - a trivial matrix $[D]$ (which changes from GP to GP only if we have heterogeneous solid)
  - a vector of external forces $[f_{ext}]$ (Neumann boundary condition)
  - Dirichlet boundary conditions are imposed either using penalty method or matrix rearrangement
- The system is solved using your preferable solver
  (see Christophe Bovet's (ONERA) lecture)

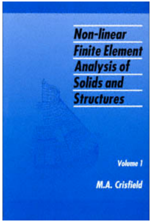
Recommended literature

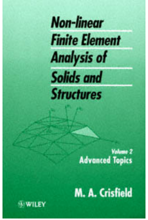












FEM from mechanical engineering prospective

Merci de votre attention !